6.897 Advanced Data Structures (Spring'05)

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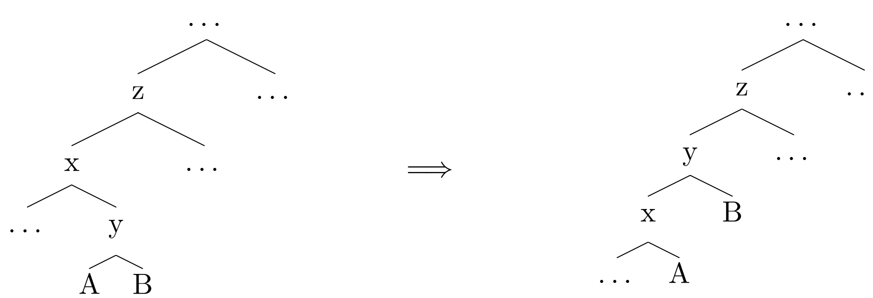
Problem 2 – Solution

Proof of the lemma. We show that we can get from any tree to a left path (any node has just a left child) with at most $n-1$ rotations. Since a rotation can be undone by another rotation, we can then get from the path to any tree with $n-1$ more rotations, so we use $\leq 2n-2$ rotations in total. (If you care to know, the optimum is $2n-6$).

The tree always has $n-1$ edges; we show that with one rotation, we can increase the number of edges on the left path starting from the root by one. So we need at most $n-1$ rotations to get all edges on that path. We proceed as follows: find a node x which branches off the left path, and rotate it up to the path. Then the edge above x becomes a left-path edge. Any other edge does not change status. See the figure:



Competitiveness with $O(\lg n)$ guarantee. We conceptually break the operations into chunks of n. We keep the count of the current operation, the cost of the current chunk, and a history of all operations ever performed. At the beginning of each chunk (the current operation is divisible by n), we configure the tree to what the α-competitive BST would look like. Finding how that BST would look like is free: we simulate the BST from the beginning of time, but this is just a conceptual step, and we're not doing any actual rotation on the real tree. Then, we configure the real BST to what it should be using $\leq 2n$ rotations. A normal operation is executed by calling the competitive BST algorithm. However, when the cost of the current chunk reaches $n \lg n$, we switch to an $O(\lg n)$ tree (say splay trees). Until the end of the chunk, we just use the splay tree algorithm, which gives $O(n \lg n)$ cost. The cost for chunk i is $O(n) + \min\{T_i, O(n \lg n)\}$, where T_i is the cost of the competitive BST for chunk i. Summing up, we get a cost of $O(n) \cdot \lceil \frac{m}{n} \rceil + \sum_i \min\{T_i, O(n \lg n)\} = O(m) + \min\{\sum_i T_i, O(m \lg n)\} = O(m) + \min\{\alpha\text{OPT}, O(m \lg n)\} = O(\min\{\alpha\text{OPT}, m \lg n\})$ – because OPT $\geq m$.