6.897
L 8.4
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Dynamic matrix inverse:

↳ maintain matrix A ($A+I$ for us)
subject to modifying row i to vector v } $O(n^2)$
modifying column j to vector v }
querying A^{-1}_{ij} & $\det A$ ← $O(1)$

- bulk update to vertex i corresponds to modifying row i & column i
- maintain A^{-1} & $\det A$ explicitly
- consider modifying column i by Δ vector: $A_{ji} \leftarrow A_{ji} + \Delta_j \forall j$
  - let $e_i^T = (0\ 0\ 0\ 0\ \underset{\uparrow}{1}\ 0\ 0\ 0\ 0)$ column i
  - $\Delta \cdot e_i^T = \begin{pmatrix}\Delta_1\\ \vdots \\ \Delta_n\end{pmatrix} (0\ 0\ 0\ 0\ \underset{\uparrow}{1}\ 0\ 0\ 0\ 0) = \begin{pmatrix} & \Delta_1 & \\ & \vdots & \\ & \Delta_n & \end{pmatrix}$ (col. i)
    → outer product

$\Rightarrow A' = A + \Delta \cdot e_i^T$

- $A' = A \cdot B$ where $B = I + \underbrace{A^{-1} \cdot \Delta}_{\text{column vector } b} \cdot e_i^T$ } $O(n^2)$ to compute
- $(A')^{-1} = B^{-1} \cdot A^{-1}$
- $B = \begin{pmatrix} 1 & & & b_1 & & \\ & 1 & & b_2 & & \\ & & 1 & \vdots & & \\ & & & 1+b_i & & \\ & & & \vdots & 1 & \\ & & & b_n & & 1 \end{pmatrix} \Rightarrow \text{adj } B = \begin{pmatrix} 1+b_i & & & -b_1 & & \\ & 1+b_i & & -b_2 & & \\ & & 1+b_i & \vdots & & \\ & & & 1 & & \\ & & & \vdots & 1+b_i & \\ & & & -b_n & & 1+b_i \end{pmatrix}$

& $\det B = 1+b_i$

$\Rightarrow B^{-1}$ computable in $O(n^2)$ (& $\det B$) & has $O(n)$ nonzeros
$\Rightarrow (A')^{-1} = B^{-1} \cdot A^{-1}$ ✓ computable in $O(n^2)$
& $\det A' = \det B \cdot \det A$