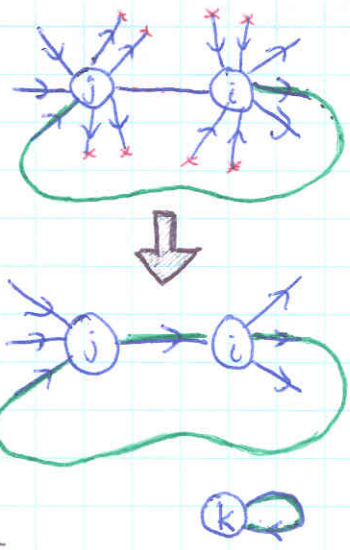
$O(n^2)$ w.c. transitive closure [Sankowski 2004]

- there is a path from i to j
$\Leftrightarrow$ after adding the edge (j,i), there is a cycle through (j,i)
$\Leftrightarrow$ after removing edges $(*,i)$ & $(j,*)$ & adding the edge (j,i), there is a cycle through i (or j)
$\Leftrightarrow$ after adding loops (k,k) & removing edges $(*,i)$ & $(j,*)$ & adding the edge (j,i), there is a disjoint set of cycles covering all vertices (disjoint cycle cover)



$\Leftrightarrow$ after the same, there is a permutation π on the vertices such that $(i, \pi(i))$ is an edge $\forall i$

Connection to linear algebra:

- adjacency matrix $A_{ij} = \begin{cases} 1 & \text{if } (i,j) \text{ is an edge} \\ 0 & \text{if not} \end{cases}$
- form $Z = (A+I)$ zap (j,i) as follows:
  - add I to A $\leftrightarrow$ add loops (k,k) (assume graph is initially loop-free)
  - zero out colum i & row j $\leftrightarrow$ remove edges $(*,i)$ & $(j,*)$
  - set Z_{ij} to 1 $\leftrightarrow$ add edge (j,i)
- $\det Z = \sum_{\pi} \text{sign}(\pi) \prod_i Z_{i\pi(i)}$

  zero $\Leftrightarrow$ any edge $(i,\pi(i))$ is missing
  - nonzero term in this sum $\Leftrightarrow$ desired permutation exists $\Leftrightarrow$ there's an edge from i to j
- adjugate: $(\text{adj } A)_{ij} = \det(A \text{ zap } (j,i))$
  - relation to inverse: $A^{-1} = \frac{\text{adj } A}{\det A}$ when it exists

$\Rightarrow$ adj $(A+I)$ computes all the terms we want to find the transitive closure
- issue remaining: cancellation between terms