Dynamic connectivity in $O(\lg^2 n)$ [Holm et al.-JACM 2001]

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High level: - store spanning forest in Euler tour tree
- hierarchically divide connected components
⇒ $\lg n$ levels of spanning forests of subgraphs, each stored in an Euler tour tree

Level of an edge is between 0 & $\lg n$ (charging mechanism)
- initially, $\lg n$
- only decreases over time (until edge is deleted)

G_i = subgraph of edges at level $\le i$ ⇒ $G_{\lg n} = G$.

Invariant 1: every connected component of G_i has $\le 2^i$ vertices

F_i = spanning forest of G_i ⇒ $F_{\lg n}$ = desired spanning forest

Invariant 2: $F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{\lg n}$ i.e. $F_i = F_{\lg n} \cap G_i$

⇒ $F_{\lg n}$ is a minimum spanning forest w.r.t. levels

Insert $(e=(v,w))$:
- add e to v & w's adjacency lists
- level(e) ← $\lg n$
- if v & w are disconnected in $F_{\lg n}$, add e to $F_{\lg n}$

$O(\lg n)$

Delete $(e=(v,w))$:
- remove e from v & w's adjacency lists
- if e is in $F_{\lg n}$:
  - delete e from F_i, $i \ge$ level(e)
  - look for replacement edge to reconnect v & w
    - can't be at level < level(e) by Invariant 2 (MSF)
    - find min. possible level ⇒ preserve Invariant 2

for i = level(e), ..., $\lg n$:
- let T_v & T_w be trees of F_i containing v & w, respectively
- relabel so that $|T_v| \le |T_w|$ ← vertex counts
- Invariant 1 ⇒ $|T_v| + |T_w| \le 2^i$ ⇒ $|T_v| \le 2^{i-1}$

⇒ can afford to push all of T_v down to level $i-1$

- for each edge (x,y) with x in T_v & level $= i$:
  - if y is in T_w: add (x,y) to $F_i, F_{i+1}, \ldots, F_{\lg n}$ & stop
  - else: level$(x,y) \leftarrow i-1$ ← charge
    ~~delete (x,y) from F_i~~