6.897
L12.3
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Matching lower bound: [Pătraşcu & Thorup – manuscript 2005]

assuming space $\le 2^{w^{1-\varepsilon}}$. (else divide w's in right 2 by a factor of a)

$$\Omega\left(\min\left\{\lg_w n, \frac{\lg w}{\lg\lg w}, \frac{\lg w}{\lg\frac{a}{\lg n}}\right\}\right)$$

Proof:

- goal: reduce $n \searrow n/2^{O(a/k)}$ & $w \searrow w/k$ as in previous upper bound
- round elim. $\Rightarrow$ $n \searrow n/bt^2$ (maybe ok) & $w \searrow w/at^2$ (way too much – need to decrease a)
- message compression lemma: [Chakrabarti & Regev – FOCS 2004]
  given a protocol for $f^{(k)}$ with m messages where Alice speaks first & with error probability p,
  there is a protocol for f with m messages where Alice speaks first & with error probability $p+\delta$ & where first msg. has length $O(a/k)/\delta^2$ (& other lengths same)
  - intuition: expected # useful bits in first msg. $= \Theta(a/k)$ so just send those
- set $\delta = \Theta(1/t)$
- $\Rightarrow$ reduce message length $a_1 \searrow O(a/k)t^2$. & reduce $w \searrow w/k$ as before with $f^{(k)} \to f$
- still can't afford round elimination from Alice to Bob
- message switching: [C & R 2004]
  given protocol for f with m msgs. where Alice speaks first & where first message has length ℓ (& error prob. δ)
  there is a protocol for f with $m-1$ msgs. where Bob's first & first msg. has length $b2^\ell$ (& error prob. δ)
  - Bob just responds to all possible msgs. from Alice
- $\Rightarrow$ eliminate Alice's msg., increase $b_1 \nearrow w \cdot 2^{\Theta(a/k)t^2}$
- round elimination lemma from Bob to Alice as before
- $\Rightarrow n \searrow n/bt^2 = n/w2^{\Theta(a/k)t^2}t^2$ → negligible
- $LB = \Omega(\min\{\lg_k w, \lg_{w2^{O(a/k)t^2}} n\})$
- maximum when $\frac{\lg w}{\lg k} = \frac{\lg n}{\lg w + \Theta(a/k)t^2} = \Theta\left(\min\left\{\frac{\lg n}{\lg w}, \frac{\lg n}{(a/k)t^2}\right\}\right)$ → fusion trees
- consider $\frac{\lg w}{\lg k} = \frac{k\lg n}{at^2} \Leftrightarrow k\lg k = \frac{at^2\lg w}{\lg n}$ → negligible: $t = O(\lg w)$

$$\Rightarrow \lg k = \lg a + 2\lg t + \lg\lg w - \lg\lg n$$

$$\Rightarrow \frac{\lg w}{\lg k} = \frac{\lg w}{\lg(a/\lg n) + \lg\lg w} = \Theta\left(\min\left\{\frac{\lg w}{\lg\frac{a}{\lg n}}, \frac{\lg w}{\lg\lg w}\right\}\right)$$