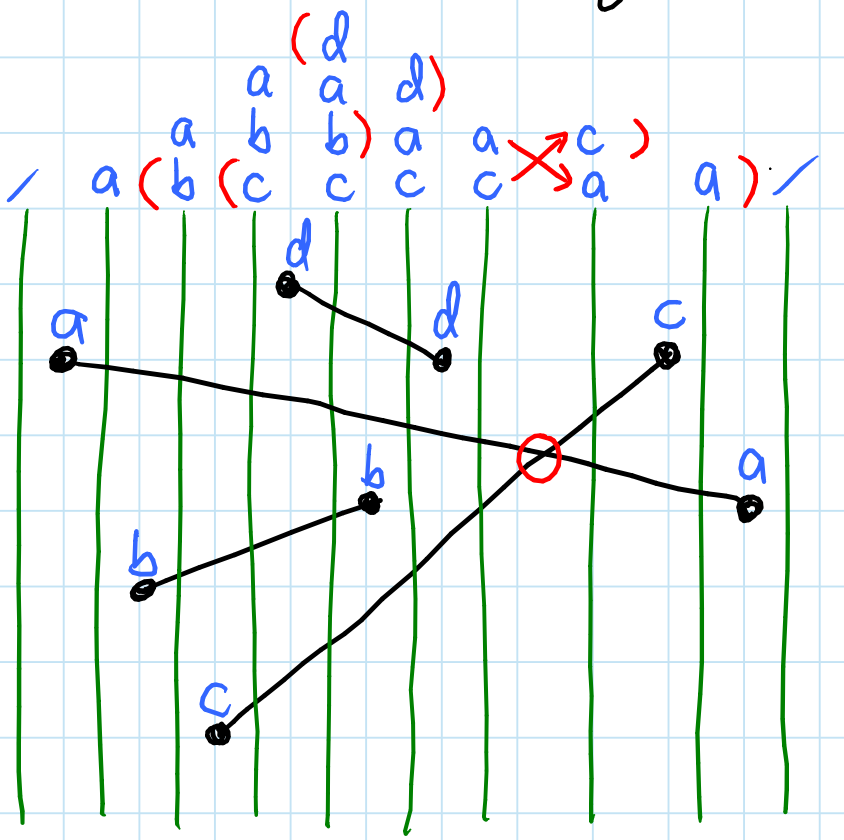
6.851	Lecture 3	Feb. 23, 2012

TODAY: Geometry I (of 2)
- point location
  - static via persistence
  - dynamic via retroactivity
- orthogonal range searching
  - range trees
  - layered range trees
- dynamization with augmentation via weight balance
- fractional cascading

Planar point location: given planar map,
(planar) graph drawn in plane with straight edges & without crossings
support query: which face contains point (x, y)?
- e.g. which GUI element got clicked?
  which city are these GPS coords. in?
- static: preprocess map
- dynamic: add/delete edges (& deg.-0 vertices)

Vertical ray shooting: given planar map, support query: which edge first hit by ray $\uparrow (x,y)$
- implies (static) solution to point location: maintain pointer from edge to face below it
- also dynamic reduction with $+O(\lg n)$ overhead

Line sweep: technique traditionally used for line-segment intersection [see e.g. CLRS]



- maintain order of intersection with vertical line which sweeps right
- left/right endpoints are inserts/deletes
- order swaps are crossings

- typically intersection DS = balanced BST

$\Rightarrow$ line-segment intersection in $O(n \lg n + k)$, where k = output size

- if we use partially persistent balanced BST then successor(y) query at time t = upward ray shooting query from (t, y)

$\Rightarrow$ $O(\lg n)$ query after $O(n \lg n)$ preprocessing [Dobkin & Lipton – SICOMP 1976]

(part of the initial motivation for persistence)

- if we use fully retroactive balanced BST then Insert/Delete(t_1, insert(y))
  + Insert/Delete(t_2, delete(y))
  = insert/delete edge $(t_1, y) \to (t_2, y)$

$\Rightarrow O(\lg n)$ dynamic vertical ray shooting among <u>horizontal</u> line segments
[Giora & Kaplan – T. Alg. 2009]
also: [Blelloch – SODA 2008] (later)

- also reduces back to retroactive successor

<u>OPEN</u>: $O(\lg n)$ dynamic vertical ray shooting in general planar map?

- $O(\lg n \lg \lg n)$ query & insert, $O(\lg^2 n)$ delete
  [Baumgarten, Jung, Mehlhorn – J. Alg. 1994]
- $O(\lg n)$ query, $O(\lg^{1+\varepsilon} n)$ insert, $O(\lg^{2+\varepsilon} n)$ delete
  [Arge, Brodal, Georgiadis – FOCS 2006]

<u>OPEN</u>: $O(\lg n)$ static ray shooting (not vertical)

- $O(\frac{n}{\sqrt{s}} \text{polylg } n)$ query & $O(s^{1+\varepsilon})$ space
  for any $1 \le s \le n$ [Agarwal – SICOMP 1992]
- conjectured nearly optimal
- 3D even harder e.g. [Agarwal & Sharir – SICOMP 1996]
- motivation: ray tracing

Orthogonal range searching:

maintain n points in d dimensions subject to
query: given box $[a_1, b_1] \times \cdots \times [a_d, b_d]$,
report existence/count/k points in box

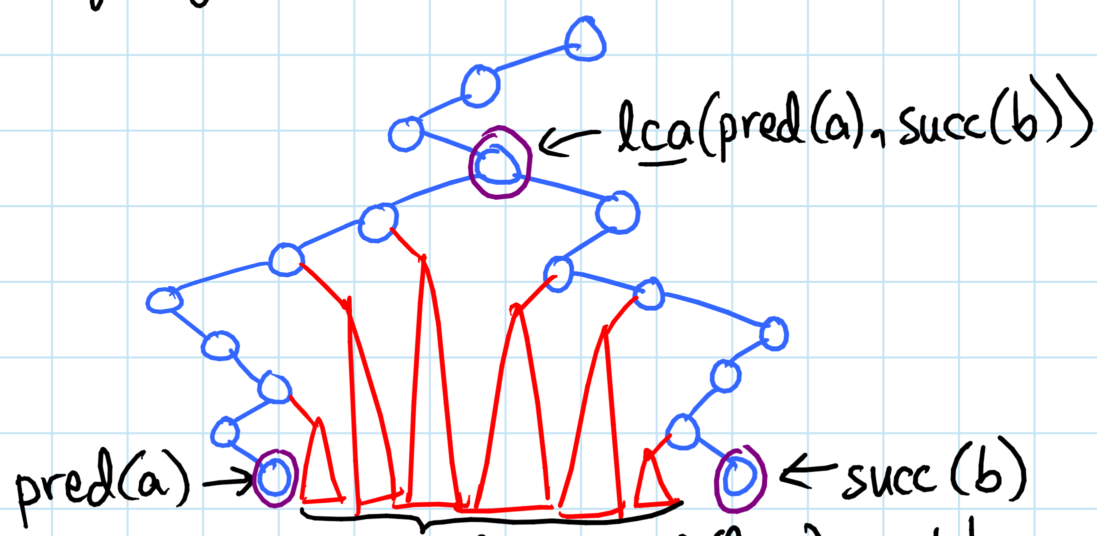
- static: preprocess points; dynamic: insert/delete
- motivation: query in database table with d cols.

Range trees: $O(\lg^d n + k)$ query

(see de Berg, Cheong, van Kreveld book)

[Bentley - IPL 1979; Lee & Wong - TDS 1980; Lueker - FOCS 1978; Willard - TR 1979]

- 1D: balanced BST on leaves = points
  - internal node key = max(left subtree)
  - query([a, b]): search(a); search(b) $\Downarrow$ BST

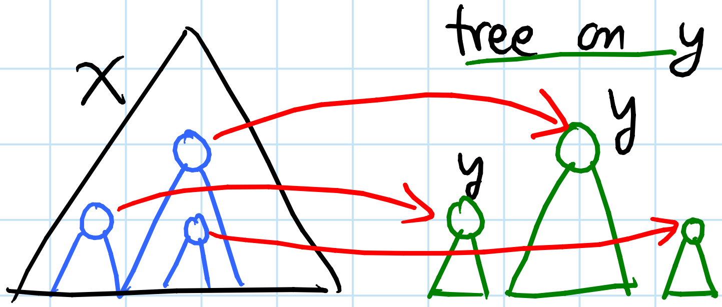


results in $O(\lg n)$ subtrees

- augment with subtree sizes to get count

(can also do this with regular BST but messier, especially to generalize)

- 2D: 1D tree on x + each subtree links to 1D tree on y on same points



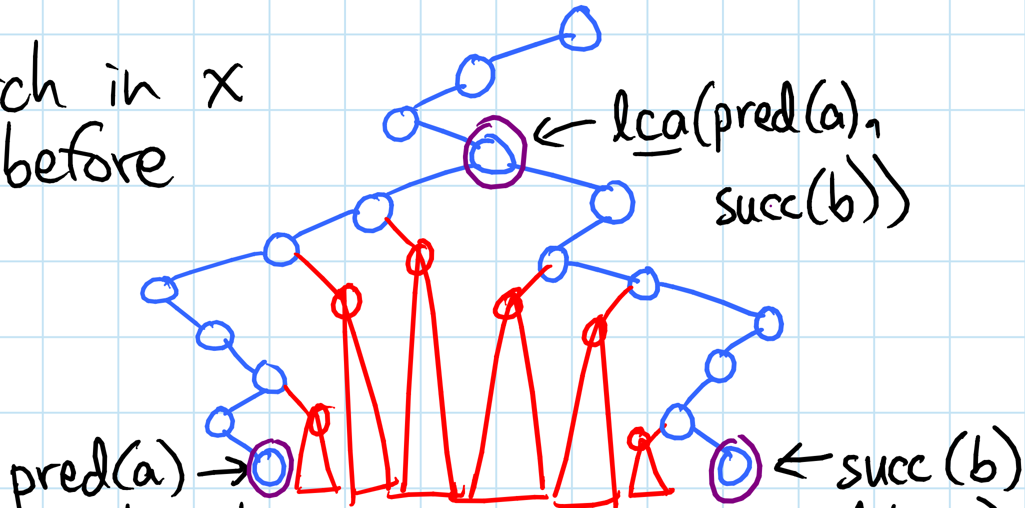
- each point appears in $\Theta(\lg n)$ structures
  $\Rightarrow \Theta(n \lg n)$ space
- query($[a_1, b_1] \times [a_2, b_2]$):
  - x query($[a_1, b_1]$) $\rightarrow O(\lg n)$ x subtrees
  - follow pointers $\rightarrow O(\lg n)$ y trees
  - $O(\lg n)$ y queries $\rightarrow O(\lg^2 n)$ y subtrees
  $\Rightarrow O(\lg^2 n + k)$ time
- augment y trees with subtree sizes for count

- dD: recurse on d
  - $O(\lg^d n)$ query
  - $O(n \lg^{d-1} n)$ space & preprocessing
  - $O(\lg^d n)$ update: recursively update each node along root-to-leaf path

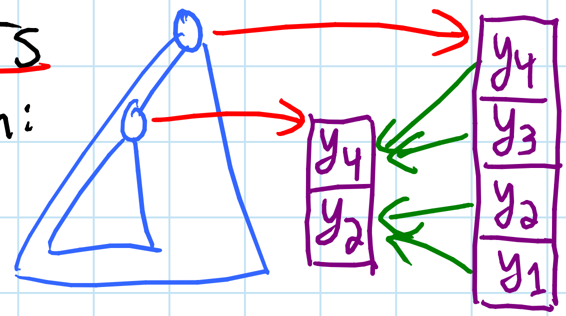
Layered range tree: $O(\lg^{d-1} n)$ query for $d > 1$

[Gabow, Bentley, Tarjan - STOC 1984; Willard - PhD 1978 / SICOMP 1985]

- 2D: search in x as before



  - store y structures as arrays (sorted by y)
  - search once in root y structure ~ $O(\lg n)$
  - carry those search results down to result subtree roots
  - from one level down: store pointers to corresponding spots (successors)



  $\Rightarrow$ find start & end in $O(\lg n)$ y arrays in $O(1)$ per level, $O(\lg n)$ overall
  - can still compute counts & report k points

- dD: same as before, just use 2D base case
  - $O(n \lg^{d-1} n)$ space & preprocessing

Dynamization with augmentation via weight balance

- BB[α] trees: [Nievergelt & Reingold — STOC 1972]
  - for each node x:
    size(left(x)) & size(right(x)) are $\geq \alpha \cdot$ size(x)
  - $\Rightarrow$ height $\leq \log_{\frac{1}{1-\alpha}} n$
  - when node is unbalanced, can afford to perfectly rebuild entire subtree of size k:
    - charge to $\Theta(k)$ of additive imbalance
    - update gets charged $\Theta(\lg n)$ times
    - $\Rightarrow O(\lg n)$ amortized cost
- applied to range tree: [idea in Lueker — FOCS 1978; Willard — SICOMP 1985]
  - rebuild costs $\Theta(k \lg^{d-1} k)$
  - $\Rightarrow O(\lg^d n)$ amortized update
  - update also updates $O(\lg n)$ y-trees $\times$ $O(\lg n)$ z-trees $\times \cdots \times O(\lg n)$ time $= O(\lg^d n)$
  - layered range trees: hard to update pointers; cost $O(\lg^d n)$ on average if array $\to$ BST at root, linked list elsewhere [Willard — SICOMP 1985]
    average case inputs

Static improvement:

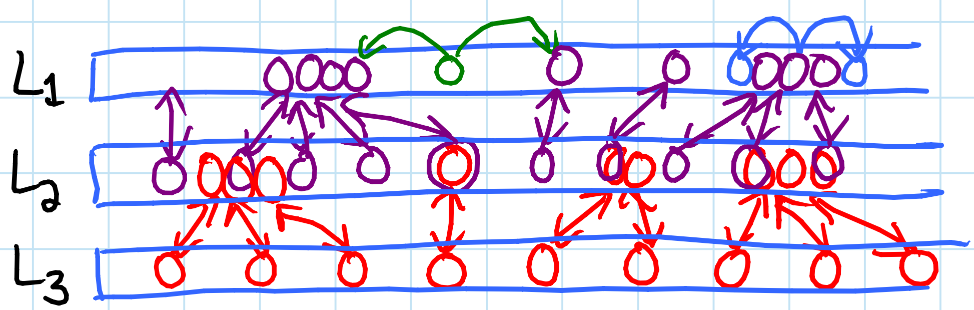
- can reduce space to $O\left(\frac{n \lg^{d-1} n}{\lg \lg n}\right)$
  [Chazelle — SICOMP 1986]
- for $d \geq 3$, can improve query to $O(\lg^{d-2} n)$
  - $O(n \lg^d n)$ space via fractional cascading
    [Chazelle & Guibas — Alg. 1986 ×2]
  - $O(n \lg^{d-1+\varepsilon} n)$ space [Alstrup, Brodal, Rauhe — FOCS 2000]

Fractional cascading: [Chazelle & Guibas – Alg. 1986 ×2]

dynamic: [Mehlhorn & Näher – Alg. 1990]

Warmup: predecessor/successor search for x "1.5D" among k lists each of length n

- $O(k \lg n)$ trivial (k binary searches)
- $O(k + \lg n)$ solution:
  - let $L_i' = L_i$ + every other element of L_{i+1}'
  - $\Rightarrow |L_i'| = n + \frac{1}{2} \cdot |L_{i+1}'| = O(n)$ (geometric)
  - link between identical elements in L_i' & L_{i+1}'
  - each element in L_i stores pointer to previous/next element in $L_i' - L_i$
  - each element in $L_i' - L_i$ stores pointer to previous/next element in L_i



- search(x):
  - binary search in L_1' → $O(\lg n)$
  - if amid $L_1' - L_1$, follow pointers to neighbors in L_1 to solve L_1 problem
  - if amid L_1, follow pointers to neighbors in $L_1' - L_1$ (else stay)
  - walk down to L_2'
  - repeat

General: graph where each
- vertex contains set of elements
- edge labeled with range [a,b]
- locally bounded degree: # incoming edges whose labels $\ni x$ is $\leq c$.

- search(x) wants to find x in k vertices' sets found by navigating (online) from any vertex, along edges whose labels $\ni x$
- improve $O(k \lg n)$ to $O(k + \lg n)$

idea: same as warmup
new: cycles in graph
but very few items go around cycles