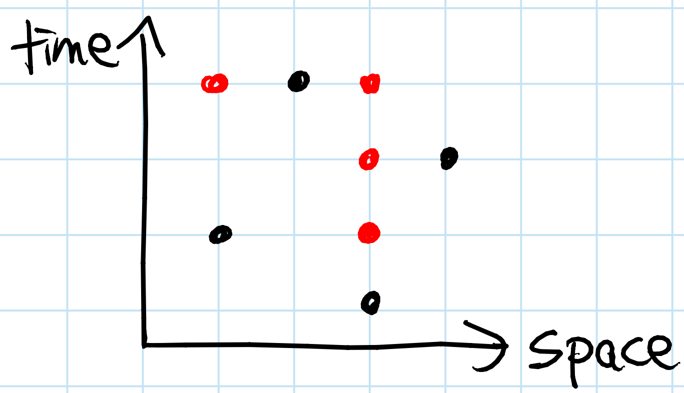
6.851	Lecture 6	Mar. 6, 2012

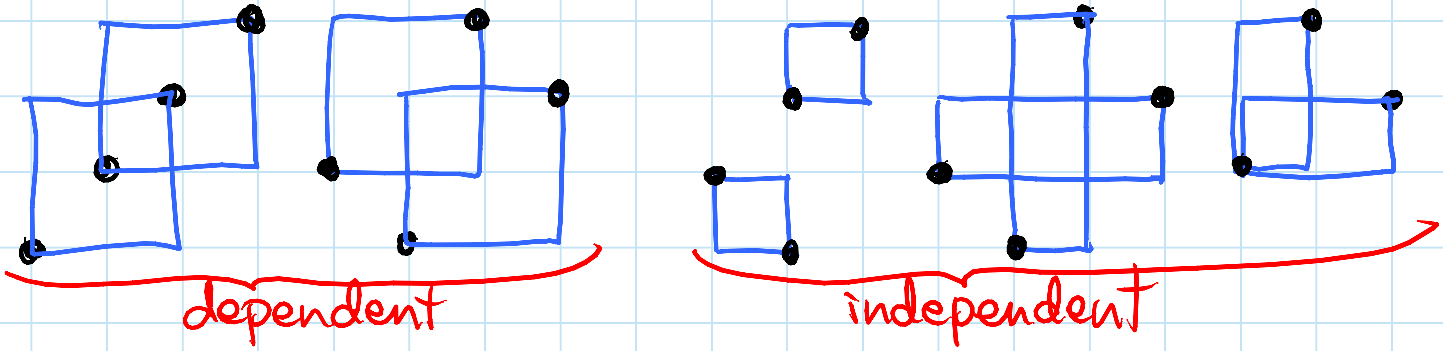
TODAY: Dynamic Optimality II (of 2)
- lower bounds:
  - independent rectangles
  - Wilber 1 & 2
  - signed greedy
- Tango trees: $O(\lg \lg n)$-competitive

Recall:
- point set is a valid BST execution
  $\Leftrightarrow$ arborally satisfied set:
  rectangle spanned by two points not on a horizontal/vertical line contains another point
- Greedy algorithm conjectured O(optimal)
- can be simulated online



Lower bounds: [Demaine, Harmon, Iacono, Kane, Pătraşcu]

Independent rectangles are unsatisfied & (⇔ in input point set (accesses))
no corner is strictly inside another

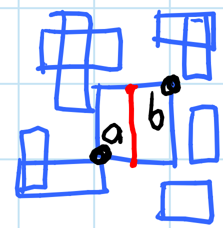


Theorem: OPT $\geq$ |input| + $\frac{1}{2}$ max # independent rectangles

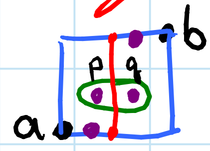
Signed rectangles: ⧄ & ⧅ types
- ⧄-satisfied if all ⧄ rectangles have another pt.
- $OPT_{⧄}$ = smallest ⧄-satisfied superset of points

Lemma: $OPT_{⧄} \geq$ |input| + max # independent ⧄-rectangles

Proof:
① find rectangle in indep. set & vertical line hitting just it
↳ segment with endpoints on top & bottom edges of rectangle

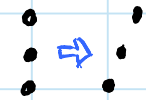


② find horizontally adjacent pts. of $OPT_{⧄}$ in rect. crossing line

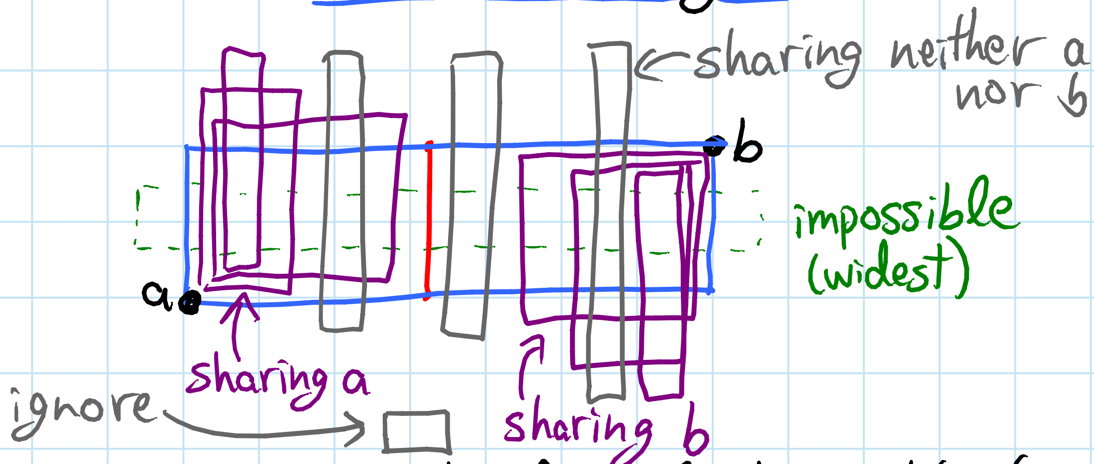


③ charge indep. rectangle to those points

Assume input x & y coords. all distinct



① : take the widest rectangle



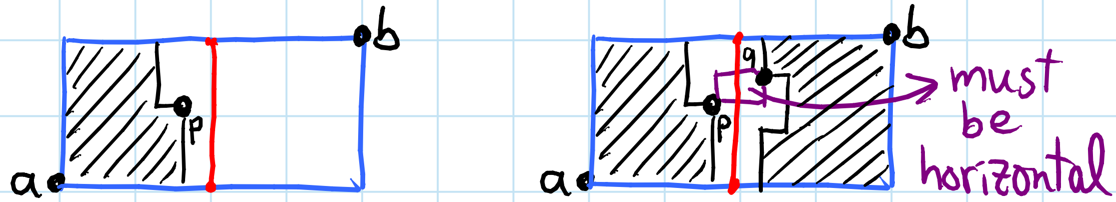
- sharing-a rects. left of sharing-b's (indep.)
- sharing-neithers fit in between vertical edges

⇒ room left for vertical line

② : take p = topmost rightmost point in rectangle & left of line (e.g. a)

q = bottommost leftmost point in rectangle & right of line & not below p (e.g. b)

∈ OPT⊠

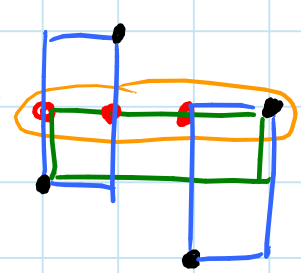


③ : p & q ($\in$ OPT⊠) are not in any other common rectangle

⇒ pair won't get charged again

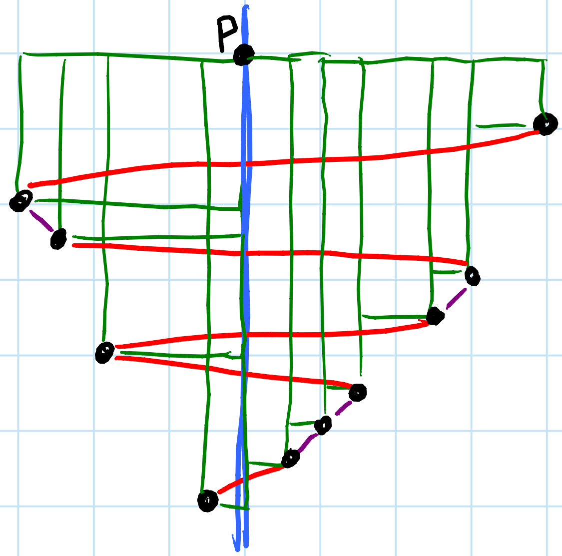
- in any horizontal chain of charges ≤1 in input (by distinct y's)

⇒ added ≥ # indep. rectangles

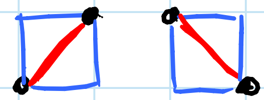


Wilber's second lower bound: [Wilber - SICOMP 1989]

(- given input (access) point set
- for each point p:
  - look at orthogonally visible points below p
  - count # alternations between left/right of p
- sum over all p



Proof: independent rectangle ∀ alternation:



Conjecture: OPT = Θ(Wilber 2)

Key-independent optimality: [Iacono - ISAAC 2002]

- suppose key values are "meaningless"

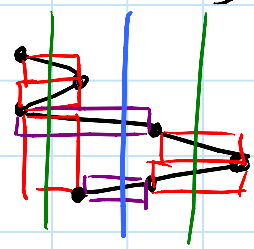
⇒ might as well permute them uniformly at random

- claim: E[OPT] = working-set bound
  - ⇒ splay trees are key-indep. optimal
- proof sketch: E[Wilber 2(x_i)] = $\Theta(\lg t_i)$

  (expected # changes to max. in random permutation)

Wilber's first lower bound: [Wilber - SICOMP 1989]

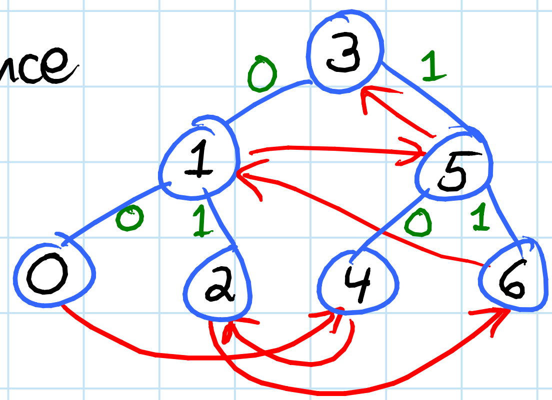
- fix a lower-bound tree P on same keys
  (e.g. perfect binary tree)
- for each node y of P:
  count # alternations in $x_1, x_2, \ldots, x_n$
  between accesses in left & right subtrees of y
  (ignoring accesses to y or outside y's subtree)
- sum over all y



Proof: independent rectangle $\forall$alternation

Example: bit-reversal sequence

000	0
001	4
010	2
011	6
100	1
101	5
110	3
111	7

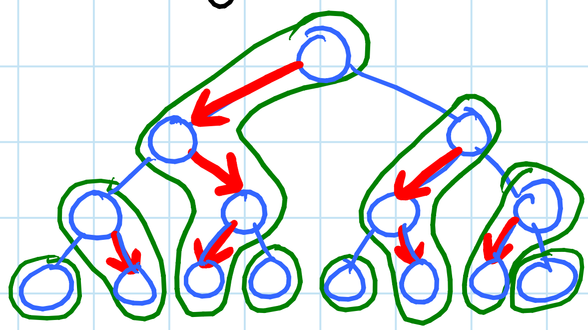


$\Rightarrow$ # alternations at y = size of y's subtree
$\Rightarrow$ Wilber 1 $= \Theta(n \lg n)$
$\Rightarrow$ OPT $= \Theta(n \lg n)$

OPEN: $\forall$ access sequence $\exists$ tree P such that
OPT $= \Theta$(Wilber 1)

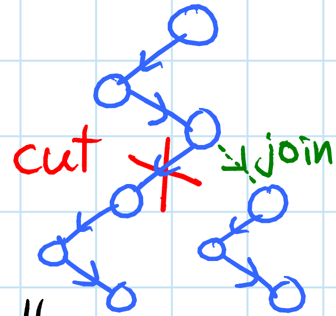
Tango trees: [Demaine, Harmon, Iacono, Pătrașcu - SICOMP 2007]

- $O(\lg \lg n)$-competitive online BST
- P = perfect BST on n keys
- define preferred child of node y in P to be
  - left if accessed left subtree of y more recently
  - right if accessed right subtree of y more recently
  - none if no access to either subtree yet
- preferred path = chain of preferred child pointers
  - partition of nodes of P

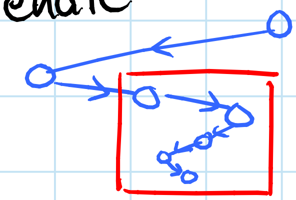


- idea: store each preferred path in auxiliary tree
  - conceptually separate balanced BST (e.g. AVL)
  - leaves link to roots of aux. trees of children paths
  - has $\leq \lg n$ nodes (height of perfect P)
  - $\Rightarrow$ supports search in $O(\lg \lg n)$ time
- search starts at top aux. tree (containing root of P)
  - each jump to next aux. tree = nonpreferred edge = preferred edge change = +1 in Wilber 1
  - k jumps $\Rightarrow$ LB k, UB $(k+1)\cdot O(\lg \lg n)$
  - $\Rightarrow O(\lg \lg n)$-competitive ... if we can update preferred edges OK

Auxiliary trees:



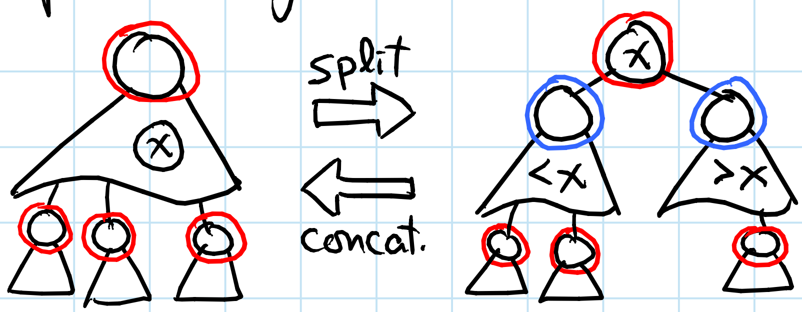
- changing a preferred child = cutting one path & joining two paths:
- if aux. trees were sorted by depth, this would be like split & concatenate
- depth $> d$ translates to interval of keys



$\Rightarrow$ can implement cuts & joins with $O(1)$ splits & concatenates
- each costs $O(\lg |\text{aux. tree}|) = O(\lg \lg n)$

In one tree: mark roots of aux. trees

- modify split & concat. to ignore children trees & manipulate adjacent trees:

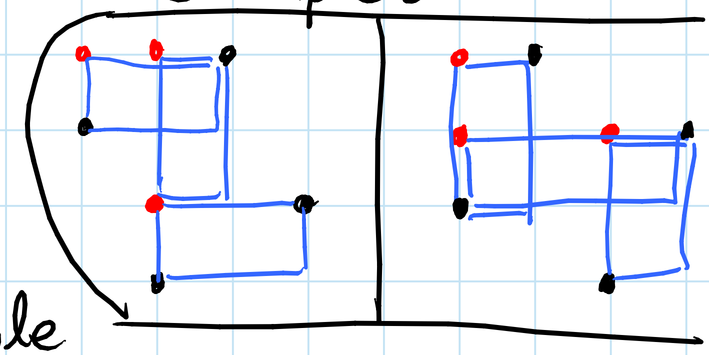


Signed Greedy:

- sweep as in Greedy
- only satisfy ⧄ boxes
- for every added point, get independent ⧄-rectangle

⇒ get lower bound: ⧄-Greedy

examples



Theorem: max{⧄-Greedy, ⧅-Greedy} = Θ(biggest independent-rectangle LB)

Proof: define $\mathrm{OPT}_{\boxtimes}$ = smallest union of ⧄-satisfying superset & ⧅-satisfying superset

$$(\mathrm{OPT} \geq)\ \mathrm{OPT}_{\boxtimes}$$
$$\geq |\text{input}| + \tfrac{1}{2} \text{ max. \# independent rectangles}$$
$$\geq \tfrac{1}{2} \max\{⧄\text{-Greedy}, ⧅\text{-Greedy}\}$$
$$\geq \tfrac{1}{2} \max\{\mathrm{OPT}_{\boxslash}, \mathrm{OPT}_{\boxbslash}\}$$
$$\geq \tfrac{1}{4} (\mathrm{OPT}_{\boxslash} + \mathrm{OPT}_{\boxbslash})$$
$$\geq \tfrac{1}{4} \mathrm{OPT}_{\boxtimes}$$

what we actually proved on p.2

⇒ constant-factor sandwich □

Summary: so close!

Greedy ⧄ & ⧅ UB	vs.	Signed Greedy ⧄ + ⧅ LB

PROJECT: compare UBs & LBs for many pt. sets