6.851	Lecture 12	Apr. 3, 2012

TODAY: Fusion trees
- sketch & why it's enough
- approximate sketch via multiplication
- parallel comparison
- most significant set bit

1 year after "cold fusion" debacle

Fusion trees: [Fredman & Willard — STOC 1990, JCSS 1993]
- store n w-bit integers — here, statically
- $O(\log_w n)$ time for predecessor/successor
- $O(n)$ space
- word RAM

$$\Rightarrow \text{predecessor} \le \min\{\underbrace{\log_w n}_{\text{fusion}}, \underbrace{\lg w}_{\text{vEB}}\} \le \sqrt{\lg n}$$

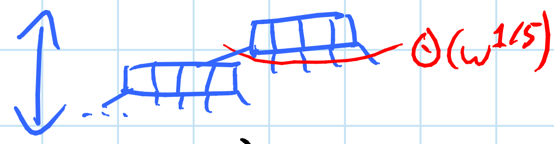
- AC^0 RAM version [Andersson, Miltersen, Thorup — TCS 1999]
  ↳ ops. are constant-depth (unbounded fan) circuits
  ⇒ no multiplication
- dynamic version via exponential trees:
  $O(\log_w n + \lg\lg n)$ deterministic updates
  [Andersson & Thorup — JACM 2007]
- dynamic version via hashing: [Raman — ESA 1996]
  $O(\log_w n)$ expected updates

- $\boxed{\text{OPEN}}$: $O(\log_w n)$ w.h.p. updates?

Idea: B-tree with branching factor $\Theta(w^{1/5})$

$\Rightarrow$ height $= \Theta(\log_w n)$

$= \Theta(\lg n / \lg w)$



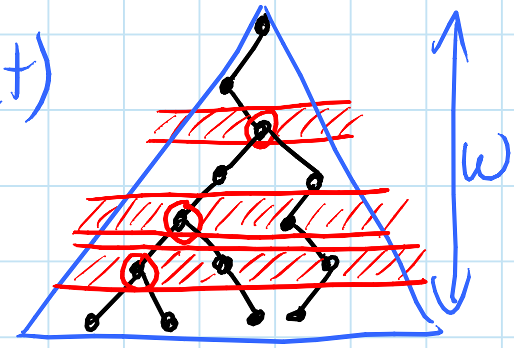
- search must visit a node in $O(1)$ time
- not enough time to read the node ($w^{1/5}$ w-bit words) to figure out which child

Fusion-tree node:

- store $k = O(w^{1/5})$ keys $x_0 < x_1 < \cdots < x_{k-1}$
- $O(1)$ time for predecessor/successor
- $k^{O(1)}$ preprocessing

Distinguishing $k = O(w^{1/5})$ keys:

- view keys $x_0, x_1, \dots, x_{k-1}$ as binary strings (0/1)
  i.e. root-to-leaf paths in
  height-w binary tree (left/right)



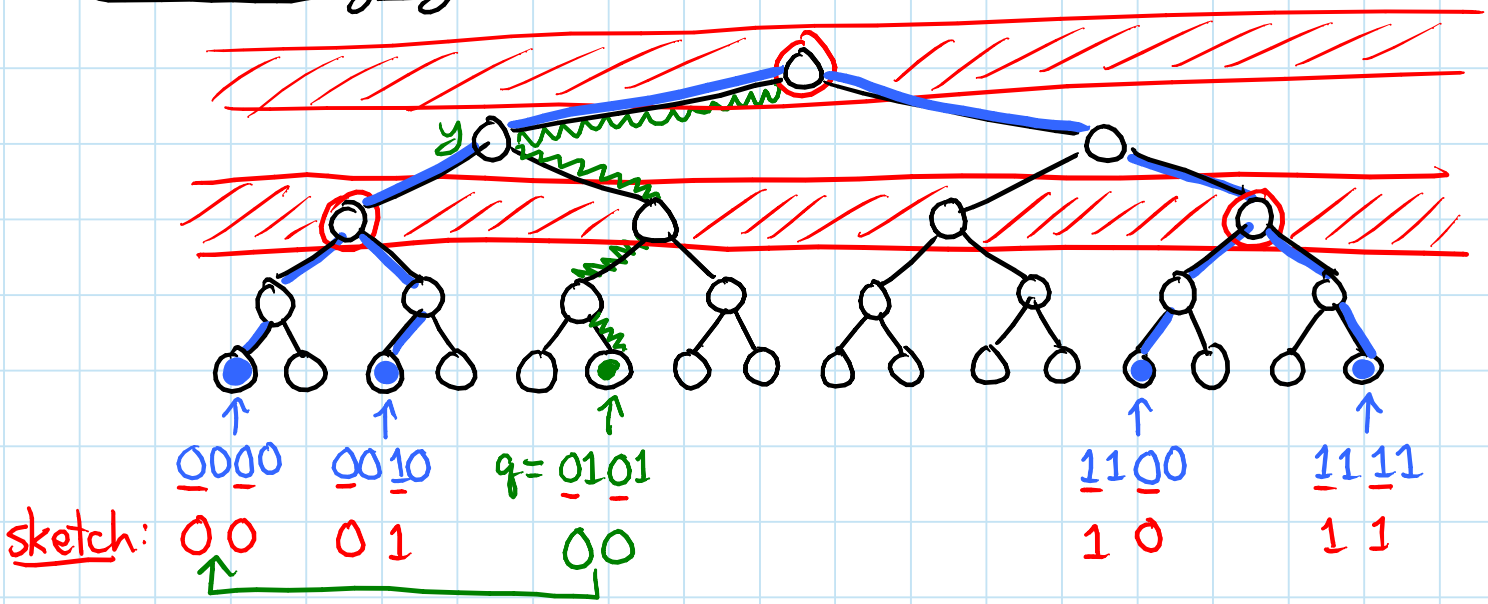
$\Rightarrow$ $k-1$ branching nodes
$\Rightarrow$ $\leq k-1$ levels
  containing branching nodes
  i.e. bits where $x_0, x_1, \dots, x_{k-1}$ first differ
  (first distinct prefix)
- call these important bits $b_0 < b_1 < \dots < b_{r-1}$,
  $r < k = O(w^{1/5})$

(perfect) sketch(x) = extract bits $b_0, b_1, \dots, b_{r-1}$ from x
  i.e. r-bit vector whose ith bit $= b_i$th bit of word x
  $\Rightarrow$ sketch(x_0) < sketch(x_1) < ⋯ < sketch(x_{k-1})
  & can pack (fuse) into one word: $k \cdot r = O(w^{2/5})$ bits
  - computable in $O(1)$ time as AC^0 operation
    [Andersson, Miltersen, Thorup – TCS 1999]
  - we'll see a cool way to compute approximate
    sketch using multiplication & standard ops.

Node search: for query q, compare sketch(q)
  in parallel to sketch(x_0), …, sketch(x_{k-1})
- again AC^0 operation on $O(1)$ words
  & we'll see a nice way with standard ops.

$\Rightarrow$ find where sketch(q) fits among sketch(x_0) < ⋯ < sketch(x_{k-1})
- want where q fits among $x_0 < \dots < x_{k-1}$

Desketchifying:



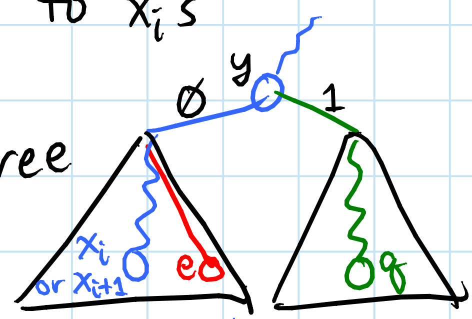
- suppose $\text{sketch}(x_i) \leq \text{sketch}(q) < \text{sketch}(x_{i+1})$
- longest common prefix = lowest common ancestor between q & (either x_i or x_{i+1})

nonsketch

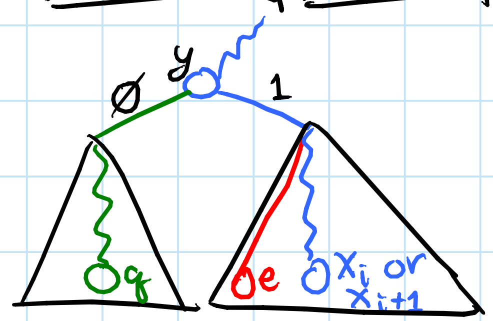
whichever's longest/lowest

= node y where q fell off paths to x_i's

- if $|y|+1$st bit of q is 1:
  - nearest x_i is in $y0$ subtree
  - nearest extreme in that subtree is $e = y011\cdots1$



- else: $e = y100\cdots0$



- predecessor & successor of q among x_i's
= predecessor & successor of sketch(e) among sketch(x_i)'s
(in terms of rank i ~ can translate to x_i)

Approximate sketch(x): on word RAM

- don't need sketch to pack b_i bits consecutively
- can spread out in predictable pattern of length $O(w^{4/5})$
  ↳ independent of x

Idea: mask important bits: $x' = x \text{ AND } \sum_{i=0}^{r-1} 2^{b_i}$

& multiply $x' \cdot m = \left(\sum_{i=0}^{r-1} x_{b_i} 2^{b_i}\right) \cdot \left(\sum_{j=0}^{r-1} 2^{m_j}\right)$

$$= \sum_{i=0}^{r-1} \sum_{j=0}^{r-1} x_{b_i} 2^{b_i + m_j}$$

Claim: for any $b_0, b_1, \ldots, b_{r-1}$, can choose $m_0, m_1, \ldots, m_{r-1}$ such that

ⓐ $b_i + m_j$ are all distinct (no collision)

ⓑ $b_0 + m_0 < \cdots < b_{r-1} + m_{r-1}$ (preserve order)

ⓒ $(b_{r-1} + m_{r-1}) - (b_0 + m_0) = O(r^4) = O(w^{4/5})$ (small)

$\Rightarrow$ approx-sketch$(x) = \left[(x \cdot m) \; \underbrace{\text{AND} \sum_{i=0}^{r-1} 2^{b_i + m_i}}_{\text{discard } i \neq j}\right] \gg (b_0 + m_0)$

Proof: ① choose $m'_0, m'_1, \ldots, m'_{r-1} < r^3$ such that $b_i + m'_j$ are all distinct modulo r^3 (strong ⓐ)

- pick $m'_0, m'_1, \ldots, m'_{t-1}$ by induction
- m'_t must avoid $\underbrace{m'_i}_{t} + \underbrace{b_j}_{r} - \underbrace{b_k}_{r}$ $\forall i, j, k$ $\Rightarrow tr^2 < r^3$ choices

$\Rightarrow$ choice for m'_t exists

for different i & j ~ if j's match, may have $b_i \equiv b_{i'} \bmod r^3$ but still $b_i \neq b_{i'}$ as needed

② let $m_i = m'_i + (w - b_i + ir^3$ rounded down to mult. of $r^3)$ ↳ to make nonnegative

$\equiv m'_i \pmod{r^3}$

$\Rightarrow m_i + b_i$ in r^3 interval after $\left(\left\lfloor \frac{w}{r^3} \right\rfloor + i\right) \cdot r^3$

$\Rightarrow \underbrace{m_0 + b_0}_{\approx w} < \cdots < \underbrace{m_{r-1} + b_{r-1}}_{\approx w + r^4}$ ⓑ

$\Rightarrow$ diff. $= O(r^4)$ ⓒ □

Parallel comparison:

protect from underflow

- $\text{sketch(node)} = 1 \ \text{sketch}(x_0) \cdots 1 \ \text{sketch}(x_{k-1})$
- $\text{sketch}(q)^k = 0 \ \text{sketch}(q) \cdots 0 \ \text{sketch}(q)$
  $= \text{sketch}(q) \cdot 0 \ 00001 \cdots 0 \ 00001$
- difference $= \binom{1}{0} \ ***** \cdots \binom{1}{0} \ *****$
- AND with $1 \ 00000 \cdots 1 \ 00000$
  $\rightarrow \binom{1}{0} \ 00000 \cdots \binom{1}{0} \ 00000$

  1 if $\text{sketch}(q) \le \text{sketch}(x_i)$
  0 if $\text{sketch}(q) > \text{sketch}(x_i)$

  $\Rightarrow$ these bits look like 0000111

  where sketch(q) fits

  need index of most sig. 1 bit

- multiply with $0 \ 00001 \cdots 0 \ 00001$
  $\rightarrow$ $\boxed{\#\,1\text{'s}}$ $\boxed{\#\,1\text{'s to right}}$ $\boxed{\text{last } 1}$

  desired

$\Rightarrow$ AND with 1111 & shift right to get # 1's
 = index of $0 \rightarrow 1$ transition
 = k-rank in sketch world

- special case of:

Index of most signifigant 1 bit:

$0\,0\,0\,\boxed{1}\,0\,1\,1\,0 \mapsto 4$
$7\,6\,5\,\boxed{4}\,3\,2\,1\,0$

- AC^0 operation [Andersson, Miltersen, Thorup 1999]
- instruction on most modern CPUs
  (see Linux kernel: include/asm-*/bitops.h ;
  GCC: __builtin_clz ; VC++: _BitScanReverse)
- needed during desketchifying (q XOR $x_{i(+1)}$)

<u>Word RAM solution:</u> [Fredman & Willard 1993]

- split word into $\sqrt{w}$ clusters of $\sqrt{w}$ bits each:

$$x = \underbrace{0101}_{\sqrt{w}} \,|\, \underbrace{0000}_{\sqrt{w}} \,|\, \underbrace{1000}_{\sqrt{w}} \,|\, \underbrace{1101}_{\sqrt{w}}$$

$\sqrt{w}$

- similar to van Emde Boas, but <u>no recursion</u>
- identify first nonempty cluster, then first 1 within

① identify nonempty clusters

- AND x with F = 1000 1000 1000 1000
  → 0000 0000 1000 1000
  = which clusters have first bit set
- XOR with x → 0101 0000 0000 0101
  = remaining bits
- subtract F − this: 0*** 1000 1000 0***
  borrow ⇔ nonempty
  no borrow ⇔ subtract ∅
- AND with F → 0000 1000 1000 0000
- XOR with F → 1000 0000 0000 1000
  nonempty
  empty
- OR with which clusters have first bit set
  → y = 1000 0000 1000 1000
  = which clusters are nonempty

② perfect sketch of y → $\underline{1}0\underline{1}\underline{1}$

- $b_i = \sqrt{w} - 1 + i\sqrt{w}$
- use $m_j = w - (\sqrt{w} - 1) - j\sqrt{w} + j$

$\Rightarrow b_i + m_j = w + (i-j)\sqrt{w} + j$ are unique for $0 \le i, j < \sqrt{w}$

& $b_i + m_i = w + i$

$\Rightarrow$ bits $w, w+1, \ldots, w+\sqrt{w}-1$ of $y \cdot m$ (shifted right w) form perfect-sketch(y)

③ find first 1 bit in sketch(y) = first nonempty cluster c

- use parallel comparison to find rank among: $\left\{ \begin{array}{l} 0001 \\ 0010 \\ 0100 \\ 1000 \end{array} \right\}$ $\sqrt{w}$ powers of 2
- fits: $\sqrt{w} \cdot (\sqrt{w} + 1) < 2w$ bits

④ find first 1 bit d in identified cluster c

- shift right $c \cdot \sqrt{w}$ & AND with 1111 to obtain cluster
- use parallel comparison as in ③

⑤ answer $= c\sqrt{w} + d$