

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Mathematics for Computer Science
MIT 6.042J/18.062J

Testing with Confidence



Albert R Meyer, April 29, 2016

testing.1

6	9	13	7
12		10	5
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98% accurate TB testing
A great-sounding diagnostic test for TB:



Albert R Meyer, April 29, 2016

testing.2

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

98% accurate TB testing
A great-sounding diagnostic test for TB: if someone has TB the test is **guaranteed** to detect it.



Albert R Meyer, April 29, 2016

testing.3

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

98% accurate TB testing
A great-sounding diagnostic test for TB: if someone has TB the test is **guaranteed** to detect it. If they don't have TB, the test says so 98% of the time.



Albert R Meyer, April 29, 2016

testing.4

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

98% accurate TB testing

$$\Pr[+ \mid \text{TB}] = 1$$

"+" for [test positive]



Albert R Meyer, April 29, 2016

testing.5

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

98% accurate TB testing

$$\Pr[+ \mid \text{TB}] = 1$$

$$\Pr[+ \mid \text{not(TB)}] = \frac{2}{100}$$

(false positive rate only 2%)



Albert R Meyer, April 29, 2016

testing.6

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

98% accurate TB testing

Overall

$$\Pr[\text{mistake}] < \frac{1}{50}$$



Albert R Meyer, April 29, 2016

testing.7

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

98% accurate TB testing

Your doctor tests you, and
it says TB!



Albert R Meyer, April 29, 2016

testing.8

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

98% accurate TB testing

Your doctor tests you, and
it says TB! He says
"The hypothesis that you
have TB holds at the **98%
confidence level.**"



Albert R Meyer, April 29, 2016

testing.9

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

98% accurate TB testing

TB is a serious disease, and
your Doc is "**98% confident**"
you have it.



Albert R Meyer, April 29, 2016

testing.10

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

98% accurate TB testing

TB is a serious disease, and
your Doc is "**98% confident**"
you have it. Should you get
treatment?



Albert R Meyer, April 29, 2016

testing.11

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Do you have TB?

TB is a serious disease, and
your Doc is "**98% confident**"
you have it. Should you get
treatment?



Albert R Meyer, April 29, 2016

testing.12

6	9	13	7
12		10	5
3	1	4	14
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Do you have TB?

TB is a serious disease, and your Doc is "98% confident" you have it. Should you get treatment? ...depends on probability you have TB.



Albert R Meyer, April 29, 2016

testing.13

6	9	13	7
12		10	5
3	1	4	14
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A Random Person

We don't really mean the "probability" that you personally have TB. We're just thinking of you as a random person.



Albert R Meyer, April 29, 2016

testing.14

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Do you have TB?

You know 2%
Pr[+ | no TB].



Albert R Meyer, April 29, 2016

testing.15

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Do you have TB?

Confidence indicates
Pr[+ | no TB].



Albert R Meyer, April 29, 2016

testing.16

6	9	13	7
12		10	5
3	1	4	14
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Confidence vs Prediction

Confidence indicates
 $\Pr[+ | \text{no TB}]$.
 You want prediction:



Albert R Meyer, April 29, 2016

testing.17

6	9	13	7
12		10	5
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Confidence vs Prediction

Confidence indicates
 $\Pr[+ | \text{no TB}]$.
 You want prediction:
 $\Pr[\text{no TB} | +]$.
 Do not confuse these!



Albert R Meyer, April 29, 2016

testing.18

6	9	13	7
12		10	5
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Confidence vs Prediction

Bayes' Theorem lets us
 find $\Pr[\text{no TB} | +]$
 given $\Pr[+ | \text{no TB}]$



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testing.19

6	9	13	7
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Odds of an Event

Event E

$$\text{Odds}[E] ::= \frac{\Pr[E]}{\Pr[\bar{E}]}$$



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testing.20

6	9	13	7
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Odds of an Event

example: 6-sided die

$$\Pr[\text{roll } 3] = \frac{1}{6}$$

$$\text{Odds}[\text{roll } 3] = \frac{1/6}{5/6} = \frac{1}{5}$$

"1 to 5"



Albert R Meyer, April 29, 2016

testing.21

6	9	13	7
12		10	5
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Odds of TB

$$\text{Odds}[\text{no TB}] = \frac{\Pr[\text{no TB}]}{\Pr[\text{TB}]}$$



Albert R Meyer, April 29, 2016

testing.22

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Do you have TB?

$$\text{Odds}[\text{no TB} | +]$$



Albert R Meyer, April 29, 2016

testing.23

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Do you have TB?

$$\text{Odds}[\text{no TB} | +] = \frac{\Pr[\text{no TB} | +]}{\Pr[\text{TB} | +]}$$

$$= \frac{\Pr[+ | \text{no TB}] \Pr[\text{no TB}] / \Pr[+]}{\Pr[+ | \text{TB}] \Pr[\text{TB}] / \Pr[+]}$$



Albert R Meyer, April 29, 2016

testing.24

6	9	13	7
12	10	5	
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Do you have TB?

$$\text{Odds}[\text{no TB} | +] = \frac{\text{Pr}[\text{no TB} | +]}{\text{Pr}[\text{TB} | +]}$$

$$= \frac{\left(\frac{\text{Pr}[+ | \text{no TB}]}{\text{Pr}[+ | \text{TB}]} \right) \left(\frac{\text{Pr}[\text{no TB}]}{\text{Pr}[\text{TB}]} \right) / \cancel{\text{Pr}[+]}}{\left(\frac{\text{Pr}[+ | \text{no TB}]}{\text{Pr}[+ | \text{TB}]} \right) \left(\frac{\text{Pr}[\text{no TB}]}{\text{Pr}[\text{TB}]} \right) / \cancel{\text{Pr}[+]}}$$



Albert R Meyer, April 29, 2016

testing.25

6	9	13	7
12	10	5	
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Do you have TB?

$$\text{Odds}[\text{no TB} | +] =$$

$$\frac{\text{Pr}[+ | \text{no TB}]}{\text{Pr}[+ | \text{TB}]} \cdot \frac{\text{Pr}[\text{no TB}]}{\text{Pr}[\text{TB}]}$$



Albert R Meyer, April 29, 2016

testing.26

6	9	13	7
12	10	5	
3	1	4	14
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Do you have TB?

$$\text{Odds}[\text{no TB} | +] =$$

$$\frac{\text{Pr}[+ | \text{no TB}]}{\text{Pr}[+ | \text{TB}]} \cdot \text{Odds}[\text{no TB}]$$

Bayes' factor



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testing.27

6	9	13	7
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Do you have TB?

$$\text{Odds}[\text{no TB} | +] =$$

$$\frac{1/50}{1} \cdot \text{Odds}[\text{no TB}]$$

Bayes' factor



Albert R Meyer, April 29, 2016

testing.28

6	9	13	7
12		10	5
3	1	4	14
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Do you have TB?

Odds[no TB | +] =

$$\frac{1}{50} \cdot \text{Odds[no TB]}$$

Bayes' factor



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testing.29

6	9	13	7
12		10	5
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11,000 TB cases reported

CDC got reports of 11,000 cases of TB in US in 2011. Will be lots of unreported.

So estimate:

$$\text{Pr}[TB] = \frac{1}{10,000}$$



Albert R Meyer, April 29, 2016

testing.30

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

11,000 TB cases reported

CDC got reports of 11,000 cases of TB in US in 2011. Will be lots of unreported.

So estimate:

$$\text{Odds[no TB]} = 9,999$$



Albert R Meyer, April 29, 2016

testing.31

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Do you have TB?

$$\text{Odds[no TB | +]} = \frac{1}{50} \cdot 9,999$$



Albert R Meyer, April 29, 2016

testing.32

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Do you have TB?

$$\text{Odds}[\text{no TB} \mid +] = 199.98$$

$$\text{Pr}[\text{no TB} \mid +] = 0.9950\dots$$



Albert R Meyer, April 29, 2016

testing.33

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Unlikely you have TB

Predicted probability of TB

$$\text{Pr}[\text{TB} \mid +] \approx \frac{1}{200}$$



Albert R Meyer, April 29, 2016

testing.34

6	9	13	7
12		10	5
3	1	4	14
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Unlikely you have TB

Because of relatively **high**
false positive rate (2%)
compared to TB rate (**0.01%**),
predicted probability of TB
remains **small (1/2 %)**!



Albert R Meyer, April 29, 2016

testing.35

6	9	13	7
12		10	5
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A "more accurate" test

98% accurate test is not so
good here.



Albert R Meyer, April 29, 2016

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

A "more accurate" test

98% accurate test is not so good here. In fact, there's a trivial test that is 99.99% accurate:

always say "No TB"



Albert R Meyer, April 29, 2016

testing.37

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

98% accuracy still useful

98% accurate test did increase your odds of TB 50 times.



Albert R Meyer, April 29, 2016

testing.38

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

98% accuracy still useful

98% accurate test did increase your odds of TB 50 times. If you only had 7M medicine doses for a population of 350M, whom should you medicate?



Albert R Meyer, April 29, 2016

testing.39

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

98% accuracy still useful

If you medicate at random you'll only medicate

$$\frac{7}{350} = 2\%$$

of sick people.



Albert R Meyer, April 29, 2016

testing.40

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

98% accuracy still useful

Instead, medicate the 7M who test positive.



Albert R Meyer, April 29, 2016

testing.41

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

98% accuracy still useful

Instead, medicate the 7M who test positive. All the sick people are sure to be among these.



Albert R Meyer, April 29, 2016

testing.42