

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Mathematics for Computer Science
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Factorials: Stirling's Formula



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stirling.1

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Closed form for $n!$

$$n! ::= 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n = \prod_{i=1}^n i$$

Turn product into a **sum** taking logs:

$$\begin{aligned} \ln(n!) &= \ln(1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n) = \\ &= \ln 1 + \ln 2 + \cdots + \ln(n-1) + \ln(n) \\ &= \sum_{i=1}^n \ln(i) \end{aligned}$$



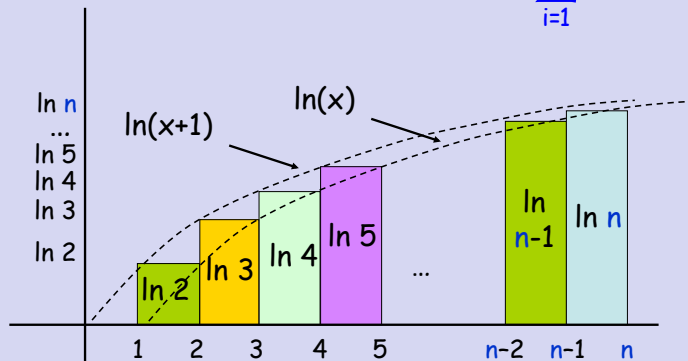
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stirling.2

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Closed form for $n!$

Integral Method to bound $\sum_{i=1}^n \ln(i)$



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Closed form for $n!$

$$n \ln\left(\frac{n}{e}\right) + 1 \leq \sum_{i=1}^n \ln(i) \leq (n+1) \ln\left(\frac{n+1}{e}\right) + 0.6$$

reminder:

$$\int \ln x \, dx = x \ln\left(\frac{x}{e}\right)$$



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Closed form for $n!$

$$\sum_{i=1}^n \ln(i) \approx \left(n + \frac{1}{2}\right) \ln\left(\frac{n}{e}\right)$$

exponentiating:

$$n! \approx \sqrt{n/e} \left(\frac{n}{e}\right)^n$$



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Stirling's Formula

A precise approximation:

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$



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stirling.6