Mathematics for Computer Science
MIT 6.042J/18.062J

Total Expectation

Law of Total Expectation

good for reasoning by cases

Def: conditional expectation

$$E[R \mid A] ::= \sum v \cdot \Pr[R = v \mid A]$$

$$E[R] = E[R \mid A] \cdot \Pr[A] + E[R \mid \overline{A}] \cdot \Pr[\overline{A}]$$

Law of Total Expectation

More generally, many cases:

$$E[R] = E[R \mid A_1] \cdot \Pr[A_1] + E[R \mid A_2] \cdot \Pr[A_2] + \cdots + E[R \mid A_n] \cdot \Pr[A_n] + \cdots$$

when $\{A_i\}$ partitions the sample space

Expected #Heads

Let $e(n)$ ::= expected #H's in n flips.

$= 1 + e(n-1)$ if 1st flip H

$= e(n-1)$ if 1st flip T

by Total Expectation:

$$e(n) = [1 + e(n-1)] \cdot p + e(n-1) \cdot q$$

$$e(n) = e(n-1)+p = e(n-2)+2p = \ldots = e(0)+np = \boxed{np}$$

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= $1 + e(n-1)$ if 1st flip H

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by Total Expectation:

$$e(n) = [1 + e(n-1)]\cdot p + e(n-1)\cdot q$$

$$e(n) = e(n-1)+p = np = E\left[B_{n,p}\right]$$