Mathematics for Computer Science
MIT 6.042J/18.062J

Linearity of Expectation

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R,S random variables, a,b constants

$$E[aR + bS] = aE[R] + bE[S]$$

even if R,S are dependent

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proof by rearranging terms in the defining sum

Linearity of Expectation

$$E[aA + bB] =$$

$$\sum_\omega (aA(\omega) + bB(\omega))\cdot\Pr[\omega] =$$

$$a(\sum_\omega A(\omega)\cdot\Pr[\omega])+b(\sum_\omega B(\omega)\cdot\Pr[\omega])$$

$$= aE[A] + bE[B]$$

QED

Expectation of indicator I_A

$$E[I_A] ::= 1 \cdot \Pr[I_A=1] + 0 \cdot \Pr[I_A=0]$$
$$= \Pr[I_A=1]$$
$$= \Pr[A]$$

Expected #Heads in n Flips

H_i is indicator for Head on i^{th} flip

$$\#H's = H_1 + H_2 + \cdots + H_n$$

Expected #Heads

$$E[\#H's] = E\left[H_1 + H_2 + \cdots + H_n\right]$$

so by linearity

$$= E[H_1] + E[H_2] + \cdots + E[H_n]$$
$$= n \cdot \Pr[\text{Head}] = np$$

Expected #hats returned

n men each check their hat. Hats get scrambled so pr[i^{th} man gets own hat back]

$$= 1/n$$

How many men do we expect will get their hat back?

Expected #hats returned

R_i indicates i^{th} man got his hat returned.

Notice R_i and R_j are not independent!

Expected #hats returned

R_i indicates i^{th} man got his hat returned. Even so

$$E[\#\text{ hats returned}] = E[R_1 + R_2 + \cdots + R_n] = E[R_1] + E[R_2] + \cdots + E[R_n]$$

$$= n(1/n) = 1$$

Chinese Banquet

Say n people sit around a spinner (a “lazy-Susan”) with n different dishes.

Spin randomly.

How many people do we expect will get same dish as initially?

Chinese Banquet

Now R_i indicates i^{th} person got same dish

R_i's are totally dependent

— all 1 or all 0

but linearity still holds

Expectation still is 1

Independent Product of Expectations

For independent X,Y

$$E[X \cdot Y] = E[X] \cdot E[Y]$$

proof by rearranging terms in the defining sum again

Independent Product of Expectations

$E[XY] ::= \sum_{x,y} xy \Pr[X=x \text{ AND } Y=y]$

(indep) $= \sum_{x,y} xy \Pr[X=x]\Pr[Y=y]$

$= \sum_y \sum_x xy \Pr[X=x]\Pr[Y=y]$

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$= E[X]E[Y]$

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Blunders

Don't assume product rule without independence.

Example: Say X takes positive and negative values with equal probability. So

$$E[X] = 0 < E[X^2]$$

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Example: Say X takes positive and negative values with equal probability. So

$$E[X] \cdot E[X] = 0 < E[X^2]$$

Blunders

Don't assume a reciprocal expectation rule. In general,

$$E\left[\frac{X}{Y}\right] \neq \frac{E[X]}{E[Y]}$$

even with independence