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Mathematics for Computer Science
MIT 6.042J/18.062J

Linearity of Expectation



Albert R Meyer,

May 8, 2013

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Linearity of Expectation

R, S random variables, a, b constants

$$E[aR + bS] = aE[R] + bE[S]$$

even if R, S are dependent



Albert R Meyer,

May 8, 2013

ranvarexpectlin.3

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Linearity of Expectation

R, S random variables, a, b constants

$$E[aR + bS] = aE[R] + bE[S]$$

proof by rearranging terms in the defining sum



Albert R Meyer,

May 8, 2013

ranvarexpectlin.4

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Linearity of Expectation

$$\begin{aligned} E[aA + bB] &= \sum_{\omega} (aA(\omega) + bB(\omega)) \cdot \Pr[\omega] \\ &= a(\sum_{\omega} A(\omega) \cdot \Pr[\omega]) + b(\sum_{\omega} B(\omega) \cdot \Pr[\omega]) \\ &= aE[A] + bE[B] \end{aligned}$$

QED



Albert R Meyer,

May 8, 2013

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Expectation of indicator I_A

$$\begin{aligned}
 E[I_A] &::= 1 \cdot \Pr[I_A=1] + \\
 &\quad 0 \cdot \Pr[I_A=0] \\
 &= \Pr[I_A=1] \\
 &= \Pr[A]
 \end{aligned}$$



Albert R Meyer,

May 8, 2013

lec.12F.6

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Expected #Heads in n Flips

H_i is indicator for
Head on i^{th} flip

$$\#H's = H_1 + H_2 + \dots + H_n$$



Albert R Meyer,

May 8, 2013

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Expected #Heads

$$\begin{aligned}
 E[\#H's] &= E[H_1 + H_2 + \dots + H_n] \\
 &\text{so by linearity} \\
 &= E[H_1] + E[H_2] + \dots + E[H_n] \\
 &= n \cdot \Pr[\text{Head}] = np
 \end{aligned}$$



Albert R Meyer,

May 8, 2013

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Expected #hats returned

n men each check their hat.
Hats get scrambled so
 $\Pr[i^{\text{th}}$ man gets own hat back]
 $= 1/n$

How many men do we expect
will get their hat back?



Albert R Meyer,

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Expected #hats returned

R_i indicates i^{th} man got his hat returned.

Notice R_i and R_j are **not** independent!



Albert R Meyer,

May 8, 2013

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Expected #hats returned

R_i indicates i^{th} man got his hat returned. Even so

$$\begin{aligned} E[\# \text{ hats returned}] &= \\ E[R_1 + R_2 + \dots + R_n] &= \\ E[R_1] + E[R_2] + \dots + E[R_n] &= \\ &= n(1/n) = 1 \end{aligned}$$



Albert R Meyer,

May 8, 2013

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Chinese Banquet

Say n people sit around a spinner (a "lazy-Susan") with n different dishes.

Spin randomly.

How many people do we **expect** will get same dish as initially?



Albert R Meyer,

May 8, 2013

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Chinese Banquet

Now R_i indicates i^{th} person got same dish

R_i 's are totally **dependent**

— all 1 or all 0

but **linearity still holds**

Expectation still is 1



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Independent Product of Expectations

For independent X, Y

$$E[X \cdot Y] = E[X] \cdot E[Y]$$

proof by rearranging terms in the defining sum again



Albert R Meyer,

May 8, 2013

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Independent Product of Expectations

$$\begin{aligned} E[XY] &::= \sum_{x,y} xy \Pr[X=x \text{ AND } Y=y] \\ (\text{indep}) &= \sum_{x,y} xy \Pr[X=x] \Pr[Y=y] \\ &= \sum_y \sum_x xy \Pr[X=x] \Pr[Y=y] \end{aligned}$$



Albert R Meyer,

May 8, 2013

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Independent Product of Expectations

$$\begin{aligned} E[XY] &::= \sum_{x,y} xy \Pr[X=x \text{ AND } Y=y] \\ (\text{indep}) &= \sum_{x,y} xy \Pr[X=x] \Pr[Y=y] \\ &= \sum_y \sum_x xy \Pr[X=x] \Pr[Y=y] \end{aligned}$$



Albert R Meyer,

May 8, 2013

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Independent Product of Expectations

$$\begin{aligned} E[XY] &::= \sum_{x,y} xy \Pr[X=x \text{ AND } Y=y] \\ (\text{indep}) &= \sum_{x,y} xy \Pr[X=x] \Pr[Y=y] \\ &= \sum_y \sum_x xy \Pr[X=x] \Pr[Y=y] \\ &= \sum_y (y \Pr[Y=y] \sum_x x \Pr[X=x]) \end{aligned}$$



Albert R Meyer,

May 8, 2013

ranvarexpectlin.20

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Independent Product of Expectations

$$\begin{aligned}
 E[XY] &::= \sum_{x,y} xy \Pr[X=x \text{ AND } Y=y] \\
 (\text{indep}) &= \sum_{x,y} xy \Pr[X=x] \Pr[Y=y] \\
 &= \sum_y \sum_x xy \Pr[X=x] \Pr[Y=y] \\
 &= \sum_y (y \Pr[Y=y] \sum_x x \Pr[X=x]) \\
 &= (\sum_x x \Pr[X=x]) (\sum_y y \Pr[Y=y]) \\
 &= E[X]E[Y]
 \end{aligned}$$



Albert R Meyer,

May 8, 2013

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Independent Product of Expectations

$$\begin{aligned}
 E[XY] &::= \sum_{x,y} xy \Pr[X=x \text{ AND } Y=y] \\
 (\text{indep}) &= \sum_{x,y} xy \Pr[X=x] \Pr[Y=y] \\
 &= \sum_y \sum_x xy \Pr[X=x] \Pr[Y=y] \\
 &= \sum_y (y \Pr[Y=y] \sum_x x \Pr[X=x]) \\
 &= (\sum_x x \Pr[X=x]) (\sum_y y \Pr[Y=y]) \\
 &= E[X]E[Y] \quad \text{QED}
 \end{aligned}$$



Albert R Meyer,

May 8, 2013

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Blunders

Don't assume product rule without independence.

Example: Say X takes positive and negative values with equal probability. So

$$E[X] = 0 < E[X^2]$$



Albert R Meyer,

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Blunders

Don't assume product rule without independence.

Example: Say X takes positive and negative values with equal probability. So

$$E[X] \cdot E[X] = 0 < E[X^2]$$



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Blunders

Don't assume a reciprocal expectation rule. In general,

$$E\left[\frac{X}{Y}\right] \neq \frac{E[X]}{E[Y]}$$

even with independence



Albert R Meyer,

May 8, 2013

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