Mathematics for Computer Science

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Introduction to Random Variables

Bigger Number Game

Guess the Bigger Number

Team 1:

- Write two integers from 0 to 7 on two pieces of paper
- Show to Team 2 face down

Team 2:

- Expose one paper and look at number
- Either stick or switch to other number

Team 2 wins if gets **larger** number

Guess the Bigger Number

Do you think one team has an advantage?

Guess the Bigger Number

Do you think one team has an advantage? Which one?

You might like to try playing the game a few times with some teammates before seeing the answers below.

Strategy for Team 2

- pick a paper to expose, giving each paper equal probability.
- if exposed number is "small" then switch, otherwise stick. That is

switch if $\leq$ threshold Z where

Z is a random integer $\in [0,7)$

Analysis of Team 2 Strategy

Let low $<$ high be the integers chosen by Team 1. There are three cases:

Analysis of Team 2 Strategy

Case M: low $\leq Z <$ high

Team 2 wins in this case, so

$\Pr[\text{Team 2 wins} \mid M] = 1$

and $\Pr[M] \geq \frac{1}{7}$

Analysis of Team 2 Strategy

Case H: high $\leq Z$

Team 2 will switch, so wins iff low card gets exposed

$\Pr[\text{Team 2 wins} \mid H] = \frac{1}{2}$

Analysis of Team 2 Strategy

Case L: $Z < \text{low}$

Team 2 will stick, so wins iff high card gets exposed

$$\Pr[\text{Team 2 wins} \mid L] = \frac{1}{2}$$

Analysis of Team 2 Strategy

So $\geq 1/7$ of time, sure win.

Rest of time, win 1/2.

By Law of Total Probability

Analysis of Team 2 Strategy

So $\geq 1/7$ of time, sure win.

Rest of time, win 1/2.

$$\Pr[\text{Team 2 wins}] =$$

$$\Pr[\text{win} \mid M] \cdot \Pr[M] +$$

$$\Pr[\text{win} \mid \overline{M}] \cdot \Pr[\overline{M}]$$

Analysis of Team 2 Strategy

So $\geq 1/7$ of time, sure win.

Rest of time, win 1/2.

$$\Pr[\text{Team 2 wins}] \geq$$

$$1 \cdot \frac{1}{7} + \frac{1}{2} \cdot \left(1 - \frac{1}{7}\right) = \boxed{\frac{4}{7}}$$

Analysis of Team 2 Strategy

So Team 2 has the advantage

Analysis of Team 2 Strategy

So Team 2 has the advantage, no matter what Team 1 does!

Team 1 Strategy

...& Team 1 can play so

$\Pr[\text{Team 2 wins}] \leq \frac{4}{7}$

no matter what

Optimal Strategy

$\Pr[\text{Team 2 wins}] = \frac{4}{7}$

is optimal for both

Random Variables

Informally: an RV is a number produced by a random process:

- threshold variable Z
- number of exposed card
- number of larger card
- number of smaller card