Problem Set 8

Due: May 11

Reading:

- Chapter 15. *Counting*, Sections 15.8. *Pigeonhole Principle* and 15.9. *Inclusion-Exclusion*
- Chapter 17. *Events and Probability Spaces*

Problem 1. (a) Show that any odd integer x in the range $10^9 < x < 2 \cdot 10^9$ containing all ten digits $0, 1, \ldots, 9$ must have consecutive even digits.

Hint: What can you conclude about the parities of the first and last digit?

(b) Show that there are 2 vertices of equal degree in any finite undirected graph with $n \geq 2$ vertices.

Hint: Cases conditioned upon the existence of a degree zero vertex.

Problem 2. (a) Let $S = \{1, 2, 3, 4\}^n$ be the set of length-n sequences $(a_1, \ldots, a_n)$ where each a_i is chosen from $\{1, 2, 3, 4\}$. What is $|S|$?

(b) How many of the sequences in S contain each of 1, 2, 3, and 4 at least once? Use Inclusion-Exclusion to find and prove your answer.

Hint: For $1 \leq i \leq 4$, define $S_i \subset S$ as the subset of sequences that do *not* contain i. What is $|S_1|$? How about $|S_1 \cap S_2|$?

Problem 3.
The results of a round robin tournament in which every two people play each other and one of them wins can be modelled a *tournament digraph*—a digraph with exactly one directed edge between each pair of distinct vertices. We'll draw a directed edge $\langle v \to w \rangle$ if player v beats player w, and otherwise we'll include directed edge $\langle w \to v \rangle$.

An n-player tournament is k-*neutral* for some $k \in [0, n)$, when, for every set of k players, there is another player who beats them all. For example, being 1-neutral is the same as not having a "best" player who beats everyone else.

This problem will prove the existence of an n-player tournament that is 10-neutral, if n is large enough. We will do this by reformulating the question in terms of probabilities. In particular, for any fixed n, we assign probabilities to each n-vertex tournament digraph by choosing a direction for the edge between any two vertices, independently and with equal probability for each edge.

(a) For any set S of 10 players, let B_S be the event that no contestant beats everyone in S. Express $\Pr[B_S]$ in terms of n.

(b) Let Q be the event that the tournament digraph is *not* 10-neutral. Prove that

$$\Pr[Q] \leq \binom{n}{10}\alpha^{n-10},$$

where $\alpha ::= 1 - (1/2)^{10}$.

Hint: Let S range over the size-10 subsets of players, so

$$Q = \bigcup_{S} B_S .$$

Use Boole's inequality.

(c) Conclude that if n is large enough, then $\Pr[Q] < 1$.

Hint: Show that the limit as n approaches infinity is 0. Why is this sufficient?

(d) Explain why the previous result implies that there is an n-player 10-neutral tournament (for a large enough $n \in \mathbb{N}$).