

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Mathematics for Computer Science  
6.042J/18.062J

# Propositional Algebra



Albert R Meyer

February 14, 2018

propositional algebra.1

6	9	13	7
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## Algebra for Equivalence

Use an algebra of equivalence to prove formulas equivalent.



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6	9	13	7
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## Proving Equivalence

Have set of equivalence rules that are **sound**: if the rules prove that two formulas are  $\equiv$ , then they really are.



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6	9	13	7
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## Proving Equivalence

and the rules are **complete**: if two formulas are  $\equiv$ , these rules can prove it.



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6	9	13	7
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**Strategy: Convert to DNF**

Come up with enough equivalence rules to convert any formula to an equivalent canonical DNF.



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6	9	13	7
12		10	5
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**Strategy: Convert to DNF**

Come up with enough equivalence rules to convert any formula to an equivalent canonical DNF. Two formulas are equiv when convert to same canonical DNF.



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6	9	13	7
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**Algebra for Equivalence**

Rules for XOR, IMPLIES

$$P \text{ IMPLIES } Q \equiv \text{NOT}(P) \text{ OR } Q$$

$$P \text{ XOR } Q \equiv (\text{NOT}(P) \text{ AND } Q) \text{ OR } (\text{NOT}(Q) \text{ AND } P)$$

Just leaves AND, OR, NOT



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6	9	13	7
12		10	5
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**Algebra for Equivalence**

Double Negation

$$\text{NOT}(\text{NOT}(P)) \equiv P$$



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6	9	13	7
12		10	5
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## Algebra for Equivalence

DeMorgan's law -AND

$$\text{NOT}(P \text{ AND } Q) \equiv$$

$$\text{NOT}(P) \text{ OR } \text{NOT}(Q)$$



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6	9	13	7
12		10	5
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## Algebra for Equivalence

DeMorgan's law -OR

$$\text{NOT}(P \text{ OR } Q) \equiv$$

$$\text{NOT}(P) \text{ AND } \text{NOT}(Q)$$

Rewrite left to right until  
NOT's only on variables



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6	9	13	7
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example

converting to a sum  
of products



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6	9	13	7
12		10	5
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move NOTs down to literals

$$\text{NOT}[\text{NOT}(P \text{ OR } Q) \text{ OR } (R \text{ AND } Q)]$$



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6	9	13	7
12		10	5
3	1	4	14
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move NOTs down to literals

NOT[NOT(P OR Q) OR (R AND Q)]  
use DeMorgan



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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

move NOTs down to literals

NOT[NOT(P OR Q) OR (R AND Q)]  
NOT[NOT(P OR Q)] AND NOT(R AND Q)



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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

move NOTs down to literals

NOT[NOT(P OR Q)] AND NOT(R AND Q)  
Double NOT



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propositional algebra.15

6	9	13	7
12		10	5
3	1	4	14
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move NOTs down to literals

NOT[NOT(P OR Q)] AND NOT(R AND Q)  
(P OR Q) AND NOT(R AND Q)



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6	9	13	7
12		10	5
3	1	4	14
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move NOTs down to literals

$(P \text{ OR } Q) \text{ AND NOT}(R \text{ AND } Q)$   
use DeMorgan



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6	9	13	7
12		10	5
3	1	4	14
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move NOTs down to literals

$(P \text{ OR } Q) \text{ AND NOT}(R \text{ AND } Q)$   
 $(P \text{ OR } Q) \text{ AND } (\bar{R} \text{ OR } \bar{Q})$



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Done move NOTs down to literals

$(P \text{ OR } Q) \text{ AND } (\bar{R} \text{ OR } \bar{Q})$



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12		10	5
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Algebra for Equivalence

Distributive Law

$P \cdot (Q + R) \equiv$   
 $(P \cdot Q) + (P \cdot R)$



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6	9	13	7
12		10	5
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## Algebra for Equivalence

### Distributive Law

$$P \text{ AND } (Q \text{ OR } R) \equiv$$

$$(P \text{ AND } Q) \text{ OR } (P \text{ AND } R)$$

Rewrite left to right until  
" OR of AND's "



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6	9	13	7
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3	1	4	14
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## Get Sum of Products

$$(P \text{ OR } Q) \text{ AND } (\bar{R} \text{ OR } \bar{Q})$$

Distribute (P OR Q)



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6	9	13	7
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## Get Sum of Products

$$(P \text{ OR } Q) \text{ AND } (\bar{R} \text{ OR } \bar{Q})$$

$$((P \text{ OR } Q) \text{ AND } \bar{R}) \text{ OR}$$

$$((P \text{ OR } Q) \text{ AND } \bar{Q})$$



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6	9	13	7
12		10	5
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## Get Sum of Products

$$((P \text{ OR } Q) \text{ AND } \bar{R}) \text{ OR}$$

$$((P \text{ OR } Q) \text{ AND } \bar{Q})$$



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propositional algebra.24

6	9	13	7
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## Get Sum of Products

$((P \text{ OR } Q) \text{ AND } \bar{R}) \text{ OR}$

$((P \text{ OR } Q) \text{ AND } \bar{Q})$

Distribute  $\bar{R}$



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6	9	13	7
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## Get Sum of Products

$((P \text{ OR } Q) \text{ AND } \bar{R}) \text{ OR}$

$((P \text{ OR } Q) \text{ AND } \bar{Q})$

$(P \text{ AND } \bar{R}) \text{ OR } (Q \text{ AND } \bar{R}) \text{ OR}$

$((P \text{ OR } Q) \text{ AND } \bar{Q})$



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propositional algebra.26

6	9	13	7
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## Get Sum of Products

$(P \text{ AND } \bar{R}) \text{ OR } (Q \text{ AND } \bar{R}) \text{ OR}$

$((P \text{ OR } Q) \text{ AND } \bar{Q})$

Distribute  $\bar{Q}$



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## Get Sum of Products

$(P \text{ AND } \bar{R}) \text{ OR } (Q \text{ AND } \bar{R}) \text{ OR}$

$((P \text{ OR } Q) \text{ AND } \bar{Q})$

$(P \text{ AND } \bar{R}) \text{ OR } (Q \text{ AND } \bar{R}) \text{ OR}$

$(P \text{ AND } \bar{Q}) \text{ OR } (Q \text{ AND } \bar{Q})$



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6	9	13	7
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Done: Sum of Products

$$(P \text{ AND } \bar{R}) \text{ OR } (Q \text{ AND } \bar{R}) \text{ OR} \\ (P \text{ AND } \bar{Q}) \text{ OR } (Q \text{ AND } \bar{Q})$$



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Simplification rules

$$Q \text{ OR } Q \equiv Q$$

$$\bar{Q} \text{ OR } Q \equiv \text{True}$$

$$Q \text{ AND } Q \equiv Q$$

$$\bar{Q} \text{ AND } Q \equiv \text{False}$$



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Simplification rules

$$Q \text{ OR } T \equiv T$$

$$Q \text{ AND } T \equiv Q$$

$$P \text{ OR } F \equiv P$$

$$P \text{ AND } F \equiv F$$



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6	9	13	7
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example

$$(P \text{ AND } \bar{R}) \text{ OR } (Q \text{ AND } \bar{R}) \text{ OR} \\ (P \text{ AND } \bar{Q}) \text{ OR } (Q \text{ AND } \bar{Q})$$

Simplify



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6	9	13	7
12		10	5
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example

$$(P \text{ AND } \bar{R}) \text{ OR } (Q \text{ AND } \bar{R}) \text{ OR} \\ (P \text{ AND } \bar{Q}) \text{ OR } (Q \text{ AND } \bar{Q})$$

Simplify



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6	9	13	7
12		10	5
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example

$$(P \text{ AND } \bar{R}) \text{ OR } (Q \text{ AND } \bar{R}) \text{ OR} \\ (P \text{ AND } \bar{Q}) \text{ OR } (\text{False})$$

Simplify



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6	9	13	7
12		10	5
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example

$$(P \text{ AND } \bar{R}) \text{ OR } (Q \text{ AND } \bar{R}) \text{ OR} \\ (P \text{ AND } \bar{Q})$$

Simplify



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we have DNF!

$$(P \text{ AND } \bar{R}) \text{ OR } (Q \text{ AND } \bar{R}) \text{ OR } (P \text{ AND } \bar{Q})$$

now to get Full DNF:



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## Full DNF for an AND-term

$(P \text{ AND } \bar{R})$       unsimplify

$(P \text{ AND } \bar{R}) \text{ AND } (Q \text{ OR } \bar{Q})$



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## Full DNF for an AND-term

$(P \text{ AND } \bar{R})$

$(P \text{ AND } \bar{R}) \text{ AND } (Q \text{ OR } \bar{Q})$       distribute

$(P \text{ AND } \bar{R} \text{ AND } Q) \text{ OR}$

$(P \text{ AND } \bar{R} \text{ AND } \bar{Q})$



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## Full DNF for an AND-term

$(P \text{ AND } \bar{R})$

$(P \text{ AND } \bar{R} \text{ AND } Q) \text{ OR}$   
 $(P \text{ AND } \bar{R} \text{ AND } \bar{Q})$       Full!



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## Rearrangement rules

$P \text{ AND } Q \equiv Q \text{ AND } P$

$(P \text{ AND } Q) \text{ AND } R \equiv$

$(P \text{ AND } Q \text{ AND } R)$

...likewise for OR



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example

$$(P \text{ AND } \bar{R} \text{ AND } Q) \text{ OR } (P \text{ AND } \bar{R} \text{ AND } \bar{Q})$$



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example

$$(P \text{ AND } \bar{R} \text{ AND } Q) \text{ OR } (P \text{ AND } \bar{R} \text{ AND } \bar{Q}) \quad \text{sort}$$



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6	9	13	7
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3	1	4	14
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example

$$(P \text{ AND } Q \text{ AND } \bar{R}) \text{ OR } (P \text{ AND } \bar{Q} \text{ AND } \bar{R})$$



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12		10	5
3	1	4	14
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Sorted Full DNF for  $(P \text{ AND } \bar{R})$

$$(P \text{ AND } Q \text{ AND } \bar{R}) \text{ OR } (P \text{ AND } \bar{Q} \text{ AND } \bar{R})$$



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6	9	13	7
12		10	5
3	1	4	14
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example

same for each AND-term,  
and OR them together:

$$\begin{aligned}
 & (P \text{ AND } Q \text{ AND } \bar{R}) \text{ OR } (P \text{ AND } Q \text{ AND } \bar{R}) \text{ OR} \\
 & (P \text{ AND } \bar{Q} \text{ AND } \bar{R}) \text{ OR } (\bar{P} \text{ AND } Q \text{ AND } \bar{R}) \text{ OR} \\
 & (P \text{ AND } \bar{Q} \text{ AND } R) \text{ OR} \\
 & (P \text{ AND } \bar{Q} \text{ AND } \bar{R})
 \end{aligned}$$



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6	9	13	7
12		10	5
3	1	4	14
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example

simplify (duplicates)

$$\begin{aligned}
 & (P \text{ AND } Q \text{ AND } \bar{R}) \text{ OR } (P \text{ AND } Q \text{ AND } \bar{R}) \text{ OR} \\
 & (P \text{ AND } \bar{Q} \text{ AND } \bar{R}) \text{ OR } (\bar{P} \text{ AND } Q \text{ AND } \bar{R}) \text{ OR} \\
 & (P \text{ AND } \bar{Q} \text{ AND } R) \text{ OR} \\
 & (P \text{ AND } \bar{Q} \text{ AND } \bar{R})
 \end{aligned}$$



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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

example

simplify (duplicates)

$$\begin{aligned}
 & (P \text{ AND } Q \text{ AND } \bar{R}) \text{ OR } \del{(P \text{ AND } Q \text{ AND } \bar{R}) \text{ OR}} \\
 & (P \text{ AND } \bar{Q} \text{ AND } \bar{R}) \text{ OR } (\bar{P} \text{ AND } Q \text{ AND } \bar{R}) \text{ OR} \\
 & (P \text{ AND } \bar{Q} \text{ AND } R) \text{ OR} \\
 & \del{(P \text{ AND } \bar{Q} \text{ AND } \bar{R})}
 \end{aligned}$$



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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

example

also sort the clauses

$$\begin{aligned}
 & (P \text{ AND } Q \text{ AND } \bar{R}) \text{ OR} \\
 & (P \text{ AND } \bar{Q} \text{ AND } \bar{R}) \text{ OR } (\bar{P} \text{ AND } Q \text{ AND } \bar{R}) \text{ OR} \\
 & (P \text{ AND } \bar{Q} \text{ AND } R)
 \end{aligned}$$



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## Canonical DNF

$(P \text{ AND } Q \text{ AND } \bar{R}) \text{ OR } (P \text{ AND } \bar{Q} \text{ AND } R) \text{ OR } (P \text{ AND } \bar{Q} \text{ AND } \bar{R}) \text{ OR } (\bar{P} \text{ AND } Q \text{ AND } \bar{R})$

Done Sorted Full DNF  
unique for each formula



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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Algebra for Equivalence

Thm. These rules are **complete**: if two formulas are  $\equiv$ , these rules can prove it.



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3	1	4	14
15	8	11	2

## Algebra for Equivalence

Because two formulas are equivalent



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12		10	5
3	1	4	14
15	8	11	2

## Algebra for Equivalence

Because two formulas are equivalent iff have same truth table



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12		10	5
3	1	4	14
15	8	11	2

## Algebra for Equivalence

Because two formulas are equivalent iff have same truth table iff have same canonical DNF.



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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Validity Checking still hard

Algebraic proofs **in general** don't beat truth tables.



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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Validity Checking still hard

Algebraic proofs in general don't beat truth tables. The **canonical DNF** is just a **copy of the truth table** as an algebraic formula.



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12		10	5
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## Validity Checking still hard

Algebraic proofs **in general** don't beat truth tables. **No efficient method** known for equivalence or validity.



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