

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Mathematics for Computer Science
MIT 6.042J/18.062J

Probabilistic Diagnosis



Albert R Meyer, May 3, 2013

bayes.1

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

99% accurate TB testing

A great-sounding diagnostic
test for TB:



Albert R Meyer, May 3, 2013

bayes.2

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

99% accurate TB testing

A great-sounding diagnostic
test for TB: if you have TB
the test is **guaranteed** to detect
it.



Albert R Meyer, May 3, 2013

bayes.3

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

99% accurate TB testing

A great-sounding diagnostic
test for TB: if you have TB
the test is **guaranteed** to detect
it. If you don't have TB, the
test says so 99% of the time.



Albert R Meyer, May 3, 2013

bayes.4

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

99% accurate TB testing

A great-sounding diagnostic test for TB: if you have TB the test is **guaranteed** to detect it. If you don't have TB, the test says so 99% of the time. Your doctor gives you the test, and **it says you have TB!**



Albert R Meyer, May 3, 2013

bayes.5

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

99% accurate TB testing

test **says TB!**

TB is a serious disease and the test is at least 99% accurate. How worried should you be? What is the probability that you actually have TB?



Albert R Meyer, May 3, 2013

bayes.6

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Do you have TB?

What is the probability that you have TB given that a 99% accurate says you do?

$$\Pr[\text{TB} \mid +] = ?$$

"+" for [test positive]



Albert R Meyer, May 3, 2013

bayes.7

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Do you have TB?

$$\Pr[+ \mid \text{TB}] = 1$$

$$\Pr[+ \mid \text{not TB}] = \frac{1}{100}$$

false positive rate only 1%



Albert R Meyer, May 3, 2013

bayes.8

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Do you have TB?

$$\Pr[\text{TB} | +] = \frac{\Pr[\text{TB AND } +]}{\Pr[+]}$$

$$= 1$$

$$= \frac{\Pr[+ | \text{TB}] \cdot \Pr[\text{TB}]}{\Pr[+]}$$



Albert R Meyer, May 3, 2013

bayes.9

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Do you have TB?

$$\Pr[\text{TB} | +] = \frac{\Pr[\text{TB AND } +]}{\Pr[+]}$$

$$= \frac{\Pr[\text{TB}]}{\Pr[+]}$$



Albert R Meyer, May 3, 2013

bayes.10

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

You do or you don't

$$\Pr[+] =$$

Total Probability Rule



Albert R Meyer, May 3, 2013

bayes.11

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

You do or you don't

$$\Pr[+] = \Pr[+ | \text{TB}] \cdot \Pr[\text{TB}]$$

$$+ \Pr[+ | \text{not TB}] \cdot \Pr[\text{not TB}]$$

Total Probability Rule



Albert R Meyer, May 3, 2013

bayes.12

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

You do or you don't

$$\begin{aligned}
 \Pr[+] &= \Pr[+ | \text{TB}] \cdot \Pr[\text{TB}] \\
 &\quad + \Pr[+ | \text{not TB}] \cdot \Pr[\text{not TB}] \\
 &= 1 \cdot \Pr[\text{TB}] \\
 &\quad + \frac{1}{100} \cdot \Pr[\text{not TB}]
 \end{aligned}$$



Albert R Meyer, May 3, 2013

bayes.13

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

You do or you don't

$$\begin{aligned}
 \Pr[+] &= \Pr[+ | \text{TB}] \cdot \Pr[\text{TB}] \\
 &\quad + \Pr[+ | \text{not TB}] \cdot \Pr[\text{not TB}] \\
 &= 1 \cdot \Pr[\text{TB}] \\
 &\quad + \frac{1}{100} \cdot (1 - \Pr[\text{TB}])
 \end{aligned}$$



Albert R Meyer, May 3, 2013

bayes.14

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Probability of Testing Positive

$$\begin{aligned}
 \Pr[+] &= \Pr[+ | \text{TB}] \cdot \Pr[\text{TB}] \\
 &\quad + \Pr[+ | \text{not TB}] \cdot \Pr[\text{not TB}] \\
 &= \frac{99}{100} \Pr[\text{TB}] + \frac{1}{100}
 \end{aligned}$$



Albert R Meyer, May 3, 2013

bayes.15

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Do you have TB?

$$\begin{aligned}
 \Pr[\text{TB} | +] &= \frac{\Pr[\text{TB}]}{\Pr[+]} \\
 &= \frac{\Pr[\text{TB}]}{\frac{99}{100} \Pr[\text{TB}] + \frac{1}{100}}
 \end{aligned}$$



Albert R Meyer, May 3, 2013

bayes.16

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Do you have TB?

$$\Pr[\text{TB} | +] = \frac{\Pr[\text{TB}]}{\Pr[+]}$$

$$= \frac{100\Pr[\text{TB}]}{99\Pr[\text{TB}] + 1}$$

What is $\Pr[\text{TB}]$?



Albert R Meyer, May 3, 2013

bayes.17

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

11,000 TB cases reported

CDC got reports of 11,000 cases of TB in US in 2011. Will be lots of unreported. So estimate:

$$\Pr[\text{TB}] \approx \frac{1}{10,000}$$



Albert R Meyer, May 3, 2013

bayes.18

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Do you have TB?

$$\Pr[\text{TB} | +] = \frac{100\Pr[\text{TB}]}{99\Pr[\text{TB}] + 1}$$

$$\approx \frac{\frac{100}{10000}}{\frac{99}{10000} + 1} \approx \frac{1}{100}$$



Albert R Meyer, May 3, 2013

bayes.19

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Unlikely you have TB

Because of relatively **high false positive rate (1%)** compared to TB rate (**0.01%**), chance of having TB remains **small (1%)!**



Albert R Meyer, May 3, 2013

bayes.21

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Unlikely you have TB

99% accurate test is not so good here.



Albert R Meyer, May 3, 2013

bayes.23

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

A "more accurate" test

99% accurate test is not so good here. In fact, there's a trivial test that is 99.99% accurate:

always say "No TB"



Albert R Meyer, May 3, 2013

bayes.24

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Bayes Rule

$$\Pr[\text{TB} | +] = \frac{\Pr[+ | \text{TB}] \cdot \Pr[\text{TB}]}{\Pr[+]}$$

$$\Pr[B | A] = \frac{\Pr[A | B] \cdot \Pr[B]}{\Pr[A]}$$



Albert R Meyer, May 3, 2013

bayes.25

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

99% accuracy still useful

99% accurate test did increase your probability of TB 100 times.



Albert R Meyer, May 3, 2013

bayes.26

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

99% accuracy still useful

99% accurate test did increase your probability of TB 100 times. If you only had 5M medicine doses for a population of 350M, whom should you medicate?



Albert R Meyer, May 3, 2013

bayes.27

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

99% accuracy still useful

Medicate the 3.5M who test positive, and you're likely to cure nearly all the cases.



Albert R Meyer, May 3, 2013

bayes.28