Mathematics for Computer Science
MIT 6.042J/18.062J

Predicate Logic, II

Validity & Soundness

Propositional Validity

True for all truth-values.

Example:

(P IMPLIES Q) OR (Q IMPLIES P)

Predicate Calculus Validity

True for all domains and predicates. *Example:*

∀z.[P(z) AND Q(z)] IMPLIES
[∀x.P(x) AND ∀y.Q(y)]

Predicate Calculus Validity

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*aka tautology

DeMorgan's Law for Quantifiers

Another valid formula:

$$\text{NOT}(\forall x.\ P(x))\ \text{IFF}\ \exists y.\ \text{NOT}(P(y))$$

DeMorgan's Law for Quantifiers

Another valid formula:

$$\text{NOT}(\text{AND}_x\ P(x))\ \text{IFF}\ \text{OR}_y\ \text{NOT}(P(y))$$

Proving Validity

$$\forall z.[P(z)\ \text{AND}\ Q(z)]\ \text{IMPLIES}\ [\forall x.P(x)\ \text{AND}\ \forall y.Q(y)]$$

Proof strategy: assume left side is T, then prove right side is T

Proving Validity

$$\forall z[Q(z) \land P(z)] \rightarrow [\forall x.Q(x) \land \forall y.P(y)]$$

Proof: Assume left hand side. That is, for all values of z in the domain, $Q(z)$ AND $P(z)$ is true. Suppose val(z) = c, an element in the domain. Then $Q(c)$ AND $P(c)$ holds, and so $Q(c)$ by itself holds. But c could have been any element of the domain. So we conclude $\forall x.Q(x)$. (by **UG**) Similarly conclude $\forall y.P(y)$. Therefore, $\forall x.Q(x)$ AND $\forall y.P(y)$ QED

Similar Example is Not Valid

$\forall z.[P(z) \text{ OR } Q(z)]$ IMPLIES

$[\forall x.P(x) \text{ OR } \forall y.Q(y)]$

Proof: Give *counter-model*, where left side of IMPLIES is T, but right side is F.

Namely, let domain ::= {1, 2},

Q(z) ::= [z = 1], P(z) ::= [z = 2].

Universal Generalization (UG)

$$\frac{F(c)}{\forall x.F(x)}$$

where c is a constant symbol that has not appeared earlier

Universal Generalization (UG)

$$\frac{F(c)}{\forall x.F(x)}$$

c is a "fresh symbol"

Universal Generalization (UG)

$$\frac{F(c)}{\forall x.F(x)}$$

Subtlety:

F(c) does not imply $\forall x.F(x)$

Universal Generalization (UG)

$$\frac{F(c)}{\forall x.F(x)}$$

…unlike propositional case instead have weaker notion of Soundness:

Universal Generalization (UG)

Weaker notion of Soundness

If $F(c)$ is valid then $\forall x.F(x)$ is valid