Conflict Midterm Exam October 12

Your name:____

	9:30AM	**Table (A–H):**	_______
Identify your Team:	**1PM**	**Table (A–K):**	_______
	2:30PM	**Table (A–E, 1–13):**	_______

- This exam is **closed book** except for a 2-sided cribsheet. Total time is 90 minutes.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem.
- In answering the following questions, you may use without proof any of the results from class or text.

DO NOT WRITE BELOW THIS LINE

Problem	Points	Grade	Grader
1	20		
2	20		
3	20		
4	20		
5	20		
Total	100		

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Problem 1 (Induction) (20 points).
Prove by induction that

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3} \tag{1}$$

for all integers $n \geq 1$. Use the equation itself as the induction hypothesis $P(n)$.

(a) Prove the

base case ($n = 1$).

(b) Now prove the

inductive step.

Problem 2 (**Binary Relations**) (**20 points**).
Five basic properties of binary relations $R : A \to B$ are:

1. R is a surjection [$\geq$ 1 in]
2. R is an injection [$\leq$ 1 in]
3. R is a function [$\leq$ 1 out]
4. R is total [$\geq$ 1 out]
5. R is empty [$=$ 0 out]

Below are some assertions about a relation R. For each assertion, write the numbers of all the properties above that the relation R must have; write "none" if R might not have any of these properties. For example, you should write "1, 4" next to the first assertion.

Variables $a, a_1, \ldots$ range over A and $b, b_1, \ldots$ range over B. You may assume A and B each have at least 2 elements.

(a) $\forall a, b.\ a\ R\ b.$ ______1, 4______

(b) $\forall a, b_1, b_2.\ [(a\ R\ b_1 \text{ AND } a\ R\ b_2) \text{ IMPLIES } b_1 = b_2].$ ______________

(c) $\exists a \forall b.\ (a\ R\ b).$ ______________

(d) $\text{NOT}[\exists a, b_1, b_2.\ (a\ R\ b_1 \text{ OR } a\ R\ b_2)].$ ______________

(e) R^{-1} is a function. ______________

(f) $A = B \text{ AND } \forall a.\ a\ R\ a.$ ______________

(g) $\exists a, b.\ a\ R\ b.$ ______________

(h) $\forall b \exists a_1.\ [a_1\ R\ b \text{ AND } (\forall a.\ a\ R\ b \text{ IMPLIES } a = a_1)].$ ______________

Problem 3 (State Machines) (20 points).
A robot named Wall-E wanders around a two-dimensional grid. He starts out at $(0, 0)$ and is allowed to take four different types of steps:

1. $(+2, -1)$
2. $(+1, -2)$
3. $(+1, +1)$
4. $(-3, 0)$

Thus, for example, Wall-E might walk as follows. The types of his steps are listed above the arrows.

$$(0,0) \xrightarrow{1} (2,-1) \xrightarrow{3} (3,0) \xrightarrow{2} (4,-2) \xrightarrow{4} (1,-2) \rightarrow \dots$$

Wall-E's true love, the fashionable and high-powered robot, Eve, awaits at $(0, 2)$.

(a) Describe a state machine model of this problem. What are the states? The transitions?

(b) Will Wall-E ever find his true love? If yes, find a path from Wall-E to Eve. If no, use the Invariant Principle to prove that no such path exists, being sure to clearly state and prove your preserved invariant. *Hint*: The value $x - y$ is not preserved, but how can it change?

Problem 4 (State Machines) (20 points).
Starting with some number of 4-cent and 7-cent stamps on the table, there are two ways to change the stamps:

(i) Add *one* 4-cent stamp, or

(ii) remove *two* 4-cent AND *two* 7-cent stamps (when this is possible).

(a) Let A be the number of 4-cent stamps, and B be the number of 7-cent stamps. The chart below indicates properties of some derived variables; fill in **EVERY CELL** with "Y" for YES or "N" for NO. (*Note*: The value of $\text{rem}(n, 2)$ is defined to be 0 when n is even, and 1 when n is odd.)

derived variables:	B	$4A + 7B$	$\text{rem}(B, 2)$	$\text{rem}(4A + 7B, 2)$
weakly increasing				
strictly increasing				
weakly decreasing				
strictly decreasing				
constant				

(b) Circle the properties below that are preserved invariants:

1. The number of 7-cent stamps (B) must be even.
2. The number of 7-cent stamps (B) must be greater than 0.
3. The total postage ($4A + 7B$) on the table must be odd.
4. $4A > 7B$.

(c) Using the Invariant Principle, show that it is impossible to have stamps with a total value of exactly 90 cents on the table when we start with no 4-cent stamps and exactly 211 of the 7-cent stamps. (You may use without proof the preserved invariance of some of the properties from part (b).)

Problem 5 (**Stable Matching**) (**20 points**).
Suppose we want to assign pairs of "buddies," who may be of the same gender, where each person has a preference rank for who they would like to be buddies with. The meaning of a *stable* buddy assignment is the same as for the familiar (gender segregated) stable matching problem. For the preference ranking given in Figure 1, show that there is no stable buddy assignment. In this figure Mergatroid's preferences aren't shown because they don't affect the outcome.

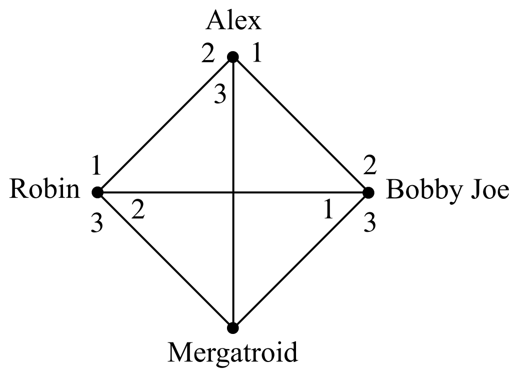

Figure 1 Some preferences with no stable buddy matching.