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12		10	5
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15	8	11	2

Mathematics for Computer Science  
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# Inclusion-Exclusion Binomial Proof



Albert R Meyer, November 12, 2017

incexbinom.1

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Incl-Excl  $n$  sets

$$|A_1 \cup A_2 \cup \dots \cup A_n| = \sum_{\emptyset \neq S \subseteq \{1,2,\dots,n\}} (-1)^{|S|+1} \left| \bigcap_{i \in S} A_i \right|$$



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Incl-Excl  $n$  sets

$$S \subseteq \{1,2,\dots,n\}$$

$$I_S ::= \bigcap_{i \in S} A_i \quad \text{for } S \neq \emptyset$$

$$I_{\emptyset} ::= A_1 \cup A_2 \cup \dots \cup A_n$$



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Incl-Excl  $n$  sets

$$|A_1 \cup A_2 \cup \dots \cup A_n| = \sum_{\emptyset \neq S \subseteq \{1,2,\dots,n\}} (-1)^{|S|+1} \left| \bigcap_{i \in S} A_i \right|$$



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## Incl-Excl $n$ sets

$$|I_{\emptyset}| = \sum_{\emptyset \neq S \subseteq \{1, 2, \dots, n\}} (-1)^{|S|+1} \left| \bigcap_{i \in S} A_i \right|$$



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## Incl-Excl $n$ sets

$$|I_{\emptyset}| = \sum_{\emptyset \neq S \subseteq \{1, 2, \dots, n\}} (-1)^{|S|+1} |I_S|$$



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## Incl-Excl $n$ sets

$$|I_{\emptyset}| = \sum_{\emptyset \neq S \subseteq \{1, 2, \dots, n\}} (-1)^{|S|+1} |I_S|$$



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## Incl-Excl $n$ sets

$$0 = \sum_{S \subseteq \{1, 2, \dots, n\}} (-1)^{|S|+1} |I_S|$$



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## Incl-Excl $n$ sets

$$\sum_{S \subseteq \{1,2,\dots,n\}} (-1)^{|S|+1} |I_S| = 0$$



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## Incl-Excl $n$ sets

$$\underbrace{\sum_{S \subseteq \{1,2,\dots,n\}} (-1)^{|S|} |I_S|}_{\sum_n} = 0$$



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## The proof

For  $a \in A_1 \cup \dots \cup A_n$

$\#a ::=$  number of times  $a$  gets counted in  $\sum_n$

**Claim:**  $\#a = 0$

so  $\sum_n = \sum_a \#a = \sum_a 0 = 0$  **QED**



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## Proof of the Claim member function

$$m_s(a) ::= \begin{cases} 1 & \text{if } a \in I_s \\ 0 & \text{otherwise} \end{cases}$$



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### Incl-Excl $n$ sets

$$\sum_{S \subseteq \{1,2,\dots,n\}} (-1)^{|S|} |I_S|$$

$$|I_S| = \sum_a m_S(a)$$



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### Incl-Excl $n$ sets

$$\sum_{S \subseteq \{1,2,\dots,n\}} (-1)^{|S|} \sum_a m_S(a)$$

distribute  $(-1)^{|S|}$

$$\sum_{S \subseteq \{1,2,\dots,n\}} \sum_a (-1)^{|S|} m_S(a)$$



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### Incl-Excl $n$ sets

$$\sum_a \sum_{S \subseteq \{1,2,\dots,n\}} (-1)^{|S|} m_S(a)$$

switch order of sums

$$\sum_{S \subseteq \{1,2,\dots,n\}} \sum_a (-1)^{|S|} m_S(a)$$



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### Incl-Excl $n$ sets

$$\sum_a \sum_{S \subseteq \{1,2,\dots,n\}} (-1)^{|S|} m_S(a)$$

#a



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## Incl-Excl $n$ sets

$$\sum_a \#a$$

$$\#a ::= \sum_{S \subseteq \{1,2,\dots,n\}} (-1)^{|S|} m_S(a)$$



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## Incl-Excl $n$ sets

break up sum by size of  $S$

$$\#a ::= \sum_{S \subseteq \{1,2,\dots,n\}} (-1)^{|S|} m_S(a)$$



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## Incl-Excl $n$ sets

$$\#a = \sum_{j=0}^n (-1)^j \sum_{|S|=j} m_S(a)$$

$$\#a ::= \sum_{S \subseteq \{1,2,\dots,n\}} (-1)^{|S|} m_S(a)$$



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## Binomial Counting proof

$$\sum_{|S|=j} m_S(a)$$

Suppose  $a$  is in exactly

$$A_2, A_3, A_4, A_8, A_9$$

then  $m_S(a) = 1$  when

$$S \subseteq \{2,3,4,8,9\}$$



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## Binomial Counting proof

$$\sum_{|S|=3} m_S(a)$$

# of such  $S$  of size 3?

$$\binom{5}{3}$$

then  $m_S(a) = 1$  when

$$S \subseteq \{2, 3, 4, 8, 9\}$$



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## Binomial Counting proof

$$\sum_{|S|=3} m_S(a) = \binom{5}{3}$$

# of such  $S$  of size 3?

then  $m_S(a) = 1$  when

$$S \subseteq \{2, 3, 4, 8, 9\}$$



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## Binomial Counting proof

$$\sum_{|S|=j} m_S(a)$$

Suppose  $a$  is in exactly  $k$  of the  $A_1, \dots, A_n$

then  $m_S(a) = 1$  when  $I_S$  consists of  $j$  these  $k$



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## Binomial Counting proof

$$\sum_{|S|=j} m_S(a) = \binom{k}{j}$$

Suppose  $a$  is in exactly  $k$  of the  $A_1, \dots, A_n$

then  $m_S(a) = 1$  when  $I_S$  is one of these  $\binom{k}{j}$  intersections



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## Binomial Counting proof

$$\#a = \sum_{j=0}^n (-1)^j \sum_{|S|=j} m_S(a)$$

$$\#a = \sum_{j=0}^k (-1)^j \binom{k}{j}$$



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## Binomial Counting proof

$$(1-1)^k = 0 \quad \text{QED}$$

$$\#a = \sum_{j=0}^k (-1)^j \binom{k}{j} =$$



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