Mathematics for Computer Science
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Inclusion-Exclusion
Binomial Proof

Incl-Excl n sets

$$|A_1 \cup A_2 \cup \cdots \cup A_n| =$$

$$\sum_{\emptyset \neq S \subseteq \{1,2,\dots,n\}} (-1)^{|S|+1} \left| \bigcap_{i \in S} A_i \right|$$

Incl-Excl n sets

$$S \subseteq \{1,2,\dots,n\}$$

$$I_S ::= \bigcap_{i \in S} A_i \quad \text{for } S \neq \emptyset$$

$$I_\emptyset ::= A_1 \cup A_2 \cup \cdots \cup A_n$$

Incl-Excl n sets

$$|A_1 \cup A_2 \cup \cdots \cup A_n| =$$

$$\sum_{\emptyset \neq S \subseteq \{1,2,\dots,n\}} (-1)^{|S|+1} \left| \bigcap_{i \in S} A_i \right|$$

Incl-Excl n sets

$$|I_{\emptyset}| = \sum_{\emptyset \neq S \subseteq \{1,2,\ldots,n\}} (-1)^{|S|+1} \left| \bigcap_{i \in S} A_i \right|$$

Incl-Excl n sets

$$|I_{\emptyset}| = \sum_{\emptyset \neq S \subseteq \{1,2,\ldots,n\}} (-1)^{|S|+1} |I_S|$$

Incl-Excl n sets

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Incl-Excl n sets

$$0 = \sum_{S \subseteq \{1,2,\ldots,n\}} (-1)^{|S|+1} |I_S|$$

Incl-Excl n sets

$$\sum_{S\subseteq\{1,2,\dots,n\}} (-1)^{|S|+1} \left|I_S\right| = 0$$

Incl-Excl n sets

$$\underbrace{\sum_{S\subseteq\{1,2,\dots,n\}} (-1)^{|S|} \left|I_S\right|}_{\sum_n} = 0$$

The proof

For $a \in A_1 \cup \cdots \cup A_n$

$\#a$ ::= number of times a gets counted in $\sum_n$

Claim: $\#a = 0$

so $\sum_n = \sum_a \#a = \sum_a 0 = 0$ QED

Proof of the Claim

member function

$$m_S(a) ::= \begin{cases} 1 \text{ if } a \in I_S \\ 0 \text{ otherwise} \end{cases}$$

Incl-Excl n sets

$$\sum_{S\subseteq\{1,2,\dots,n\}} (-1)^{|S|} |I_S|$$

$$|I_S| = \sum_a m_S(a)$$

Incl-Excl n sets

$$\sum_{S\subseteq\{1,2,\dots,n\}} (-1)^{|S|} \sum_a m_S(a)$$

distribute $(-1)^{|S|}$

$$\sum_{S\subseteq\{1,2,\dots,n\}} \sum_a (-1)^{|S|} m_S(a)$$

Incl-Excl n sets

$$\sum_a \sum_{S\subseteq\{1,2,\dots,n\}} (-1)^{|S|} m_S(a)$$

switch order of sums

$$\sum_{S\subseteq\{1,2,\dots,n\}} \sum_a (-1)^{|S|} m_S(a)$$

Incl-Excl n sets

$$\sum_a \sum_{S\subseteq\{1,2,\dots,n\}} (-1)^{|S|} m_S(a)$$

#a

Incl-Excl n sets

$$\sum_a \#a$$

$$\#a ::= \sum_{S \subseteq \{1,2,\ldots,n\}} (-1)^{|S|} m_S(a)$$

Incl-Excl n sets

break up sum by size of S

$$\#a ::= \sum_{S \subseteq \{1,2,\ldots,n\}} (-1)^{|S|} m_S(a)$$

Incl-Excl n sets

$$\#a = \sum_{j=0}^{n} (-1)^j \sum_{|S|=j} m_S(a)$$

$$\#a ::= \sum_{S \subseteq \{1,2,\ldots,n\}} (-1)^{|S|} m_S(a)$$

Binomial Counting proof

$$\sum_{|S|=j} m_S(a)$$

Suppose a is in exactly

A_2, A_3, A_4, A_8, A_9

then $m_S(a) = 1$ when

$$S \subseteq \{2,3,4,8,9\}$$

Binomial Counting proof

$$\sum_{|S|=3} m_S(a)$$

# of such S of size 3? $\binom{5}{3}$

then $m_S(a) = 1$ when

$$S \subseteq \{2,3,4,8,9\}$$

Binomial Counting proof

$$\sum_{|S|=3} m_S(a) = \binom{5}{3}$$

# of such S of size 3?

then $m_S(a) = 1$ when

$$S \subseteq \{2,3,4,8,9\}$$

Binomial Counting proof

$$\sum_{|S|=j} m_S(a)$$

Suppose a is in exactly k of the $A_1,\ldots,A_n$

then $m_S(a) = 1$ when I_S consists of j these k

Binomial Counting proof

$$\sum_{|S|=j} m_S(a) = \binom{k}{j}$$

Suppose a is in exactly k of the $A_1,\ldots,A_n$

then $m_S(a) = 1$ when I_S is one of these $\binom{k}{j}$ intersections

Binomial Counting proof

$$\#a = \sum_{j=0}^{n} (-1)^j \sum_{|S|=j} m_S(a)$$

$$\#a = \sum_{j=0}^{k} (-1)^j \binom{k}{j}$$

Binomial Counting proof

$$(1-1)^k = 0$$ QED

$$\#a = \sum_{j=0}^{k} (-1)^j \binom{k}{j} =$$