

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Mathematics for Computer Science
MIT 6.042J/18.062J

Min-Gray Edges give Min-Weight Tree



Albert R Meyer October 31, 2017

min-gray.1

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Connected Graph G

Let G be a **connected**,
weighted simple graph.
Assume all edges have
different weights.



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min-gray.2

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Black-white coloring

Let G be a **connected**,
weighted simple graph.
Color each vertex of G
black or **white**



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min-gray.3

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Black-white coloring

Let G be a **connected**,
weighted simple graph.
Color each vertex of G
black or **white**, not all
same color.



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min-gray.4

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Gray Edges

A **gray edge** connects vertices with different colors:



There must be a gray edge since G connected



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min-gray.5

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Gray Edges

A **gray edge** connects vertices with different colors:



Let e be a **min-weight** gray edge.



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min-gray.6

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Min Gray Edge **Necessary**

Theorem:

e is an edge of **every** **min-weight** spanning tree (**MST**)



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min-gray.7

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Min Gray Edges **Sufficient**

There is a spanning tree built of min-weight gray edges
-- from previous slides.



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min-gray.8

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Unique **MST**

Corollary: There is a unique MST.



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min-gray.9

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

min-weight gray = **MST**

Corollary: There is a unique MST. It consists of all min-weight gray edges under black-white colorings.



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min-gray.10

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Gray Edge **Swap** Lemma

Let C be a **connected spanning subgraph (css)** of G .



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min-gray.11

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Gray Edge **Swap** Lemma

Suppose e **not** an edge of **css** C .



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min-gray.12

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Gray Edge Swap Lemma

Suppose e not an edge of CSS C . Then there is an edge g of C :

- (i) $wt(e) < wt(g)$
- (ii) $C - g + e$ is a CSS



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min-gray.13

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Gray Edge Swap Lemma

So C is not minimum CSS because $C - g + e$ has smaller weight.



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min-gray.14

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Min Gray Edge Necessary

So e is necessarily in any min-weight CSS .



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min-gray.15

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Proof of Swap Lemma

Say $e = \langle u - v \rangle$.

C connected, so have path

$$\vec{p} = u - \dots - v.$$

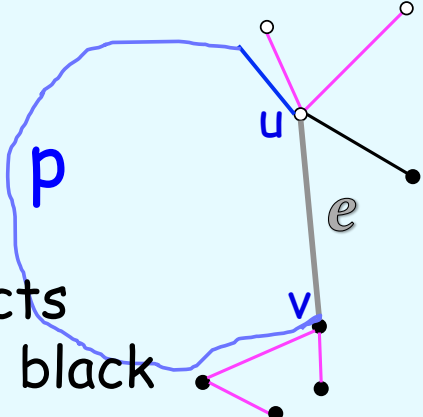


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min-gray.16

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Proof of Swap Lemma



p connects white to black

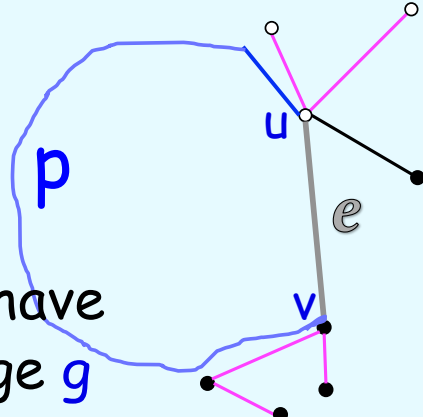
CC BY SA

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min-gray.17

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Proof of Swap Lemma



p must have gray edge **g**

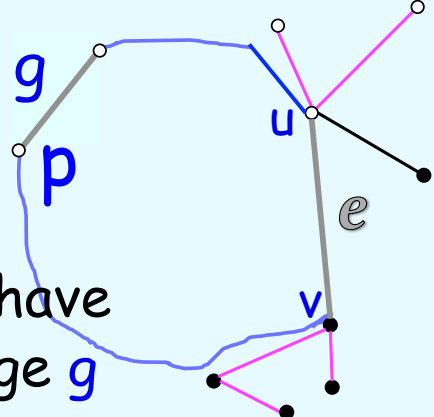
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min-gray.18

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Proof of Swap Lemma



p must have gray edge **g**

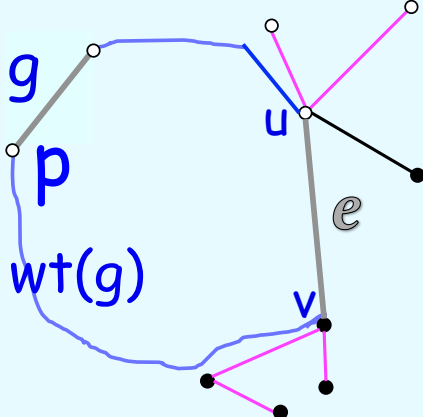
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min-gray.19

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Proof of Swap Lemma



$wt(e) < wt(g)$

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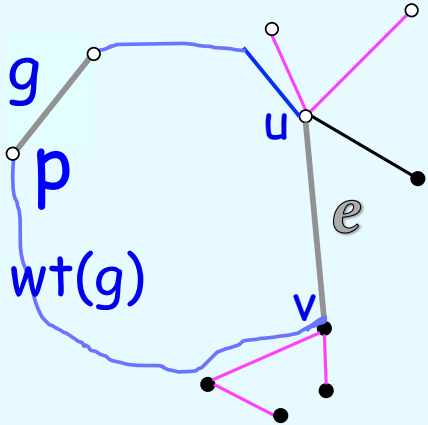
min-gray.20

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Proof of Swap Lemma

$C - g$ g

$wt(e) < wt(g)$



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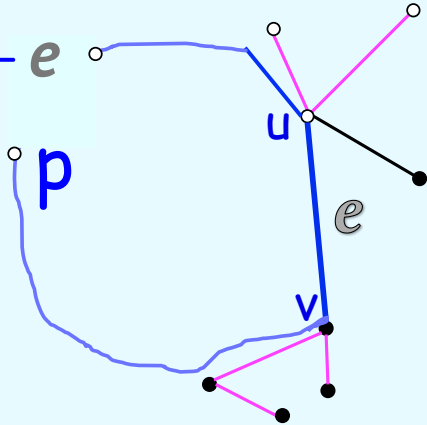
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min-gray.21

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Proof of Swap Lemma

$C - g + e$



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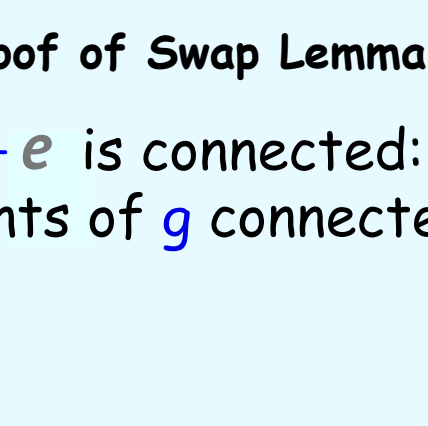
min-gray.22

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Proof of Swap Lemma

$C - g + e$ is connected:
end-points of g connected
by path

path



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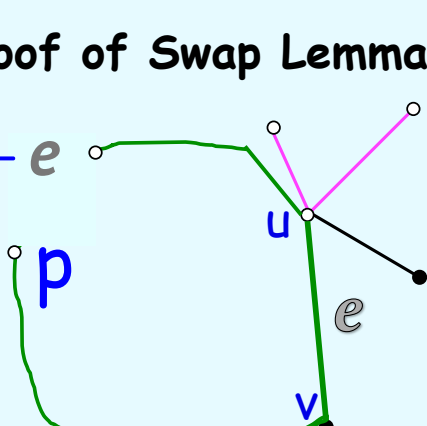
min-gray.23

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Proof of Swap Lemma

$C - g + e$

QED



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min-gray.24