

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Mathematics for Computer Science  
MIT 6.042J/18.062J

# Sums & Money



Albert R Meyer, April 10, 2013

geometric-sum.1

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Geometric Series

$$G_n = 1 + x + x^2 + \dots + x^n$$

$$-xG_n = -x - x^2 - \dots - x^n - x^{n+1}$$



Albert R Meyer, April 10, 2013

geometric-sum.2

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Geometric Series

$$G_n = 1 + x + x^2 + \dots + x^n$$

$$-xG_n = -x - x^2 - \dots - x^n - x^{n+1}$$



Albert R Meyer, April 10, 2013

geometric-sum.3

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Geometric Series

$$G_n = 1 + x + x^2 + \dots + x^n$$

$$-xG_n = -x - x^2 - \dots - x^n - x^{n+1}$$



Albert R Meyer, April 10, 2013

geometric-sum.4

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

## Geometric Series

$$\begin{array}{r}
 G_n = 1 + x + x^2 + \dots + x^n \\
 -xG_n = -x - x^2 - \dots - x^n - x^{n+1} \\
 \hline
 1 \qquad \qquad \qquad -x^{n+1}
 \end{array}$$



Albert R Meyer,

April 10, 2013

geometric-sum.5

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

## Geometric Series

$$\begin{array}{r}
 G_n - xG_n = \\
 \\
 1 \qquad \qquad \qquad -x^{n+1}
 \end{array}$$



Albert R Meyer,

April 10, 2013

geometric-sum.6

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

## Geometric Series

$$G_n - xG_n = 1 - x^{n+1}$$

$$G_n = \frac{1 - x^{n+1}}{1 - x}$$



Albert R Meyer,

April 10, 2013

geometric-sum.7

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

## Geometric Series

$$G_n = \frac{1 - x^{n+1}}{1 - x}$$

Consider *infinite* sum (series)

$$1 + x + x^2 + \dots + x^{n-1} + x^n + \dots = \sum_{i=0}^{\infty} x^i$$



Albert R Meyer,

April 10, 2013

geometric-sum.8

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

## Infinite Geometric Series

$$G_n = \frac{1 - x^{n+1}}{1 - x}$$

$$\lim_{n \rightarrow \infty} G_n = \frac{1 - \lim_{n \rightarrow \infty} x^{n+1}}{1 - x} = \frac{1}{1 - x}$$



Albert R Meyer,

April 10, 2013

geometric-sum.9

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

## Infinite Geometric Series

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1 - x}$$

for  $|x| < 1$



Albert R Meyer,

April 10, 2013

geometric-sum.10

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

## The future value of \$\$

I will pay you \$100 in 1 year,  
if you will pay me \$X now.



Albert R Meyer,

April 10, 2013

geometric-sum.11

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

## The future value of \$\$

My bank will pay me 3% interest.  
define *bankrate*

$$b ::= 1.03$$

— bank increases my \$\$ by  
this factor in 1 year.



Albert R Meyer,

April 10, 2013

geometric-sum.12

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## The future value of \$\$

If I deposit your  $\$X$  now,  
I will have  $\$b \cdot X$  in 1 year.  
So I won't lose money as long as

$$b \cdot X \geq 100$$

$$X \geq \$100/1.03 \approx \$97.09$$



Albert R Meyer,

April 10, 2013

geometric-sum.13

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## The future value of \$\$

- $\$1$  in 1 year is worth  $\$0.9709$  now.
- $\$r$  last year is worth  $\$1$  today,  
where  $r ::= 1/b$ .
- So  $\$n$  paid in 2 years is worth  
 $\$nr$  paid in 1 year, and is worth  
 $\$nr^2$  today.



Albert R Meyer,

April 10, 2013

geometric-sum.14

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## The future value of \$\$

$\$n$  paid  $k$  years from now  
is worth  $\$n \cdot r^k$  today  
where  $r ::= 1/\text{bankrate}$ .



Albert R Meyer,

April 10, 2013

geometric-sum.15

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Annuities

I pay you  $\$100/\text{year}$  for 10 years,  
if you will pay me  $\$Y$  now.

I can't lose if you pay me

$$100r + 100r^2 + 100r^3 + \dots + 100r^{10}$$

$$= 100r(1 + r + \dots + r^9)$$

$$= 100r(1 - r^{10}) / (1 - r) = \$853.02$$



Albert R Meyer,

April 10, 2013

geometric-sum.16

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

## Annuities

I pay you \$100/year for 10 years,  
if you will pay me \$853.02.

**QUICKIE:** If bankrates unexpectedly increase in the next few years,

- A. You come out ahead
- B. The deal stays fair
- C. I come out ahead



Albert R Meyer,

April 10, 2013

geometric-sum.17

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

## Manipulating Sums

$$\frac{d}{dx} \left( \sum_{i=0}^n x^i \right) = \frac{d}{dx} \left( \frac{1-x^{n+1}}{1-x} \right)$$

$$\sum_{i=0}^n ix^{i-1} = \frac{1}{x} \sum_{i=1}^n ix^i = \frac{d}{dx} \left( \frac{1-x^{n+1}}{1-x} \right)$$



Albert R Meyer,

April 10, 2013

geometric-sum.18

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

## Manipulating Sums

$$\sum_{i=1}^n ix^{i-1} = \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2}$$



Albert R Meyer,

April 10, 2013

geometric-sum.19