

6	9	13	7
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Mathematics for Computer Science
MIT 6.042J/18.062J

Gambler's Ruin

Probability of Winning



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Gambler's Ruin

- Place \$1 bets until going broke or hitting target
- What is $\Pr[\text{hit target}]$?

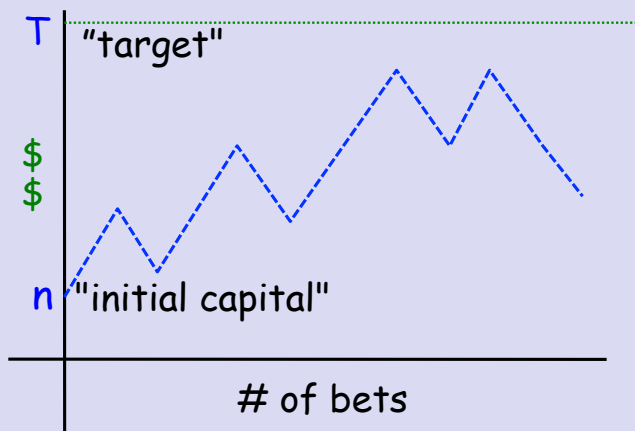


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Gambler's Ruin



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Dow Jones Trend



random steps with "up" bias?



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Gambling: Fair Case

Suppose we're playing a **fair game**:

$$\Pr[\text{win bet}] = 1/2$$

$\Pr[\text{hit } \$200]$ starting
with $\$100$? $1/2$

$\Pr[\text{hit } \$600]$ starting
with $\$500$? $5/6$



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Gambling: Fair Case

For **fair game** in general

$$\Pr[\text{hit } \$T] = \frac{n}{T}$$

What about an **unfair** game?



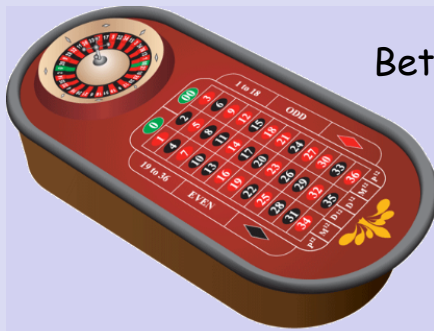
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Gambling: Slightly Unfair



Betting **black** in
US roulette

$$\Pr[\text{win bet}] = 18/38 = 9/19 < 1/2$$



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US Roulette

What is $\Pr[\text{hit } \$500 + 100]$ starting
with $\$500$? ($5/6$ when fair)

$$< 1 / 37,000$$

What is $\Pr[\text{reach } \$1M + 100]$ starting
with $\$1M$? (≈ 1 when fair)

$$< 1 / 37,000$$

no matter how many $\$$ at start!



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Gambler's Ruin

Parameters

$p ::= \text{Pr}[\text{win } \$1 \text{ bet}]$

$n ::= \text{initial capital}$

$T ::= \text{gambler's target}$

What is $\text{Pr}[\text{hit target}]?$



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Condition on 1st bet

$w_n ::= \text{Pr}[\text{hit } \$T \text{ from } \$n]$

$\text{Pr}[w_n \mid \text{win 1st bet}] = w_{n+1}$

$\text{Pr}[w_n \mid \text{lose 1st bet}] = w_{n-1}$

$$w_n = w_{n+1} \cdot p + w_{n-1} \cdot q$$



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A Linear Recurrence

$$w_{n+1} = (1/p)w_n - (q/p)w_{n-1}$$

$w_0 = 0$ (Gambler is broke)

$w_T = 1$ (Gambler at target)

Solve as usual and get:



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Linear Recurrence

$$w_n = \frac{r^n - 1}{r - 1} \cdot w_1$$

Twist: we don't know w_1

for $r ::= q/p \neq 1$.

But $w_T = 1$, so can solve for w_1 . Then

$$w_n = \frac{r^n - 1}{r^T - 1}$$



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Winning when Biased Against

$$w_n = \frac{r^n - 1}{r^T - 1} < \frac{r^n}{r^T} \text{ intended profit}$$

$$= \left(\frac{1}{r} \right)^{T-n}$$

Suppose $p < q$, so $r ::= q/p > 1$.



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Winning when Biased Against

$$w_n < (1/r)^{\text{intended profit}}$$

w_n bound does not depend on n !

$1/r < 1$, so w_n is exponentially decreasing in intended profit!



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Profit \$100 in US Roulette

$$p = 18/38 \quad q = 20/38 \quad 1/r = 9/10$$

$$\Pr[\text{profit } \$100] < (9/10)^{100}$$

$$< 1/37,648$$



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Profit \$200 in US Roulette

$$p = 18/38 \quad q = 20/38 \quad 1/r = 9/10$$

$$\Pr[\text{profit } \$200] < (9/10)^{200}$$

$$< (1/37,648)^2$$

$$< 1/70,000,000$$



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What About the Fair Case?

$$w_n = \frac{r^n - 1}{r^T - 1} \quad (r ::= q/p = 1)$$

Uh oh, dividing by 0.

Use l'Hôpital's Rule

$$\lim_{r \rightarrow 1} \frac{d(r^n - 1)/dr}{d(r^T - 1)/dr} = \frac{nr^{n-1}}{Tr^{T-1}} = \frac{n}{T}$$



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...otherwise Gambler is ruined

Claim:

$$\Pr[\text{ruin}] = 1 - \Pr[\text{win}]$$

so if we know one, we know the other.



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...otherwise Gambler is ruined

because $\Pr[\text{play forever}] = 0$:
with any stake

$$\Pr[\text{ends in } < T \text{ bets}] > \varepsilon$$

so

$$\Pr[\geq kT \text{ bets}] < (1-\varepsilon)^k$$



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