Mathematics for Computer Science

MIT 6.042J/18.062J

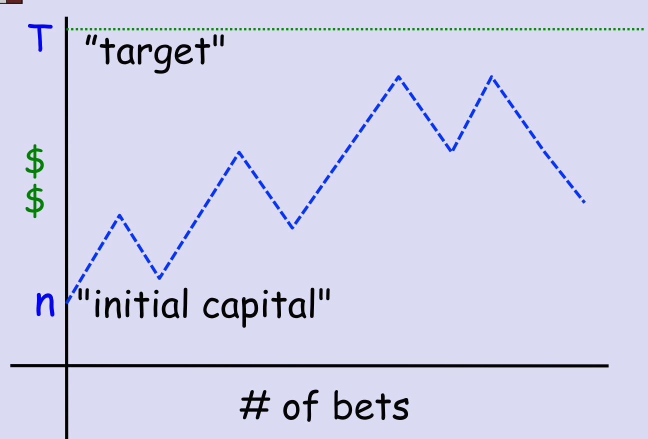
Gambler's Ruin

Probability of Winning

Gambler's Ruin

- Place $1 bets until going broke or hitting target
- What is Pr[hit target]?

Gambler's Ruin



Dow Jones Trend

random steps with "up" bias?

Gambling: Fair Case

Suppose we're playing a fair game:

Pr[win bet] = 1/2

Pr[hit $200] starting with $100? **1/2**

Pr[hit $600] starting with $500? **5/6**

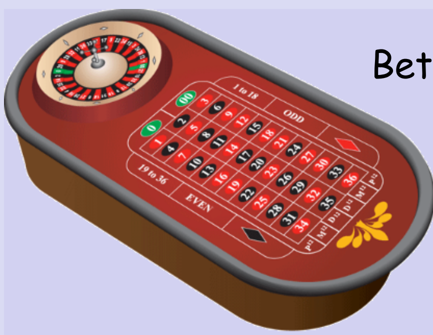
Gambling: Fair Case

For fair game in general

$$\Pr[\text{hit } \$T] = \frac{n}{T}$$

What about an unfair game?

Gambling: Slightly Unfair



Betting **black** in US roulette

Pr[win bet] = 18/38 = 9/19 < 1/2

US Roulette

What is Pr[hit $500 + 100] starting with $500? (5/6 when fair)

< 1 / 37,000

What is Pr[reach $1M + 100] starting with $1M? (≈ 1 when fair)

< 1 / 37,000

no matter how many $ at start!

Gambler's Ruin

Parameters

p ::= Pr[win \$1 bet]

n ::= initial capital

T ::= gambler's target

What is Pr[hit target]?

Condition on 1st bet

w_n ::= Pr[hit \$T from \$n]

$\Pr[w_n \mid \text{win 1st bet}] = w_{n+1}$

$\Pr[w_n \mid \text{lose 1st bet}] = w_{n-1}$

$$w_n = w_{n+1} \cdot p + w_{n-1} \cdot q$$

A Linear Recurrence

$$w_{n+1} = (1/p)w_n - (q/p)w_{n-1}$$

$w_0 = 0$ (Gambler is broke)

$w_T = 1$ (Gambler at target)

Solve as usual and get:

Linear Recurrence

$$w_n = \frac{r^n - 1}{r - 1} \cdot w_1$$

Twist: we don't know w_1

for $r ::= q/p \neq 1$.

But $w_T = 1$, so can solve for w_1. Then

$$w_n = \frac{r^n - 1}{r^T - 1}$$

Winning when Biased Against

$$w_n = \frac{r^n - 1}{r^T - 1} < \frac{r^n}{r^T} = \left(\frac{1}{r}\right)^{T-n}$$

intended profit: $T-n$

Suppose $p < q$, so $r ::= q/p > 1$.

Winning when Biased Against

$$w_n < (1/r)^{\text{intended profit}}$$

w_n bound does not depend on n!

$1/r < 1$, so w_n is exponentially decreasing in intended profit!

Profit \$100 in US Roulette

$p = 18/38$ $q = 20/38$ $1/r = 9/10$

$$\Pr[\text{profit } \$100] < (9/10)^{100}$$

$$< 1/37{,}648$$

Profit \$200 in US Roulette

$p = 18/38$ $q = 20/38$ $1/r = 9/10$

$$\Pr[\text{profit } \$200] < (9/10)^{200}$$

$$< (1/37{,}648)^2$$

$$< 1/70{,}000{,}000$$

What About the Fair Case?

$$w_n = \frac{r^n - 1}{r^T - 1} \quad (r ::= q/p = 1)$$

Uh oh, dividing by 0.

Use l'Hôpital's Rule

$$\lim_{r \to 1} \frac{d(r^n-1)/dr}{d(r^T-1)/dr} = \frac{nr^{n-1}}{Tr^{T-1}} = \frac{n}{T}$$

...otherwise Gambler is ruined

Claim:

$$\Pr[\text{ruin}] = 1 - \Pr[\text{win}]$$

so if we know one, we know the other.

...otherwise Gambler is ruined

because Pr[play forever] = 0 :

with any stake

$$\Pr[\text{ends in} < T \text{ bets}] > \varepsilon$$

so

$$\Pr[\geq kT \text{ bets}] < (1-\varepsilon)^k$$