Conflict Final 1

Your name:____________________________________

- This exam is **closed book** except for two 2-sided cribsheets. Total time is 180 minutes.
- Write your solutions in the space provided. If you need more space, **write on the back** of the sheet containing the problem.
- In answering the following questions, you may use without proof any of the results from class or text.

DO NOT WRITE BELOW THIS LINE

Problem	Points	Grade	Grader
1	12		
2	16		
3	14		
4	18		
5	14		
6	20		
7	16		
8	15		
9	15		
10	25		
11	35		
Total	200		

Short-Answer Questions

The following questions are short-answer. The graders will not read explanations, so do not spend time including them.

Problem 1 (Quantifiers) (12 points).
Let $f : \mathbb{N} \to \mathbb{R}$ be a real-valued total function. A *limit point* of f is a real number $r \in \mathbb{R}$ such that $f(n)$ is close to r for *infinitely many* n, where "close to" means within distance ϵ for whatever positive real number ϵ you may choose.

We can express the fact that r is a limit point of f with a logical formula of the form:

$$\mathbf{Q}_0\ \mathbf{Q}_1\ \mathbf{Q}_2.\ |f(n) - r| \leq \epsilon,$$

where $\mathbf{Q}_0, \mathbf{Q}_1, \mathbf{Q}_2$ is a sequence of three quantifiers from among:

$$\begin{array}{cccc}
\forall n, & \exists n, & \forall n \geq n_0, & \exists n \geq n_0. \\
\forall n_0, & \exists n_0, & \forall n_0 \geq n, & \exists n_0 \geq n. \\
\forall \epsilon \geq 0, & \exists \epsilon \geq 0, & \forall \epsilon > 0, & \exists \epsilon > 0.
\end{array}$$

Here the n, n_0 range over nonnegative integers, and ϵ ranges over real numbers.

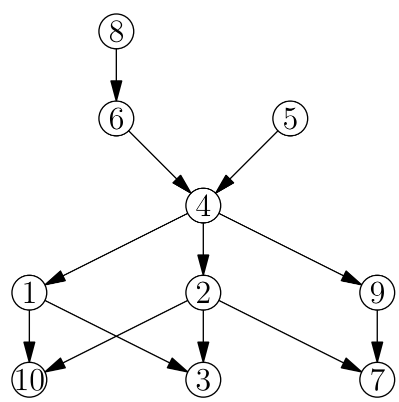
Identify the proper quantifers:

$\mathbf{Q}_0$ __________

$\mathbf{Q}_1$ __________

$\mathbf{Q}_2$ __________

Problem 2 (Scheduling) (16 points).
The following DAG describes the prerequisites among unit-time tasks $\{1, \ldots, 10\}$.



(a) What is the minimum parallel time to complete all the tasks?

(b) List a maximal antichain in this partial order.

(c) What is the minimum parallel time if no more than two tasks can be completed in parallel?

(d) How many maximal antichains are there?

Problem 3 (**Cardinality**) (**14 points**).
Find an example of sets A and B such that $\mathbb{N}$ strict A strict B.

A =

B =

Problem 4 (**Simple graphs; Counting**) (**18 points**).
Answer the following questions about **finite simple graphs**. You may answer with formulas involving exponents, binomial coefficents, and factorials.

(a) How many edges are there in the *complete graph* K_{14}? _______

(b) How many edges are there in a spanning tree of K_{14}? _______

(c) What is the chromatic number $\chi(K_{14})$? _______

(d) What is the chromatic number $\chi(C_{14})$, of the cycle of length 14? _______

(e) What is the largest possible number of leaves possible in a tree with 14 vertices? _______

(f) What is the largest number of degree-two vertices possible in a tree with 14 vertices? _______

Problem 5 (**Expectation**) (**14 points**).
In this problem you will check a proof of:

Theorem (Murphy's Law)**.** *Let $A_1, A_2, \dots A_n$ be mutually independent events, and let T be the number of these events that occur. The probability that none of the events occur is at most $e^{-\text{Ex}[T]}$.*

To prove Murphy's Law, note that

$$T = T_1 + T_2 + \cdots + T_n, \tag{1}$$

where T_i is the indicator variable for the event A_i.

For each line of the following proof, write the number from the list below of the item that justifies the line.

Proof.

$$
\begin{aligned}
\Pr[T = 0] &= \overline{A_1 \cup A_2 \cup \cdots \cup A_n} && \underline{\text{(i)}} \\
&= \Pr[\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}] && \underline{\qquad\quad} \\
&= \prod_{i=1}^{n} \Pr[\overline{A_i}] && \underline{\qquad\quad} \\
&= \prod_{i=1}^{n} 1 - \Pr[A_i] && \underline{\qquad\quad} \\
&\le \prod_{i=1}^{n} e^{-\Pr[A_i]} && \underline{\qquad\quad} \\
&= e^{-\sum_{i=1}^{n} \Pr[A_i]} && \underline{\qquad\quad} \\
&= e^{-\sum_{i=1}^{n} \operatorname{Ex}[T_i]} && \underline{\qquad\quad} \\
&= e^{-\operatorname{Ex}[T]}. && \underline{\qquad\quad}
\end{aligned}
$$

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Justification

(i) def. of $[T = 0]$

(ii) Union bound

(iii) pairwise independence

(iv) mutual independence

(v) linearity of Ex[]

(vi) expectation of an indicator

(vii) distributivity

(viii) De Morgan's law

(ix) complement rule

(x) Chebyshev inequality

(xi) $1 - x \le e^{-x}$ for all x

(xii) $1 + x \le e^{x}$ for $|x| \le 1$

(xiii) $1 + x = o(e^{x})$

(xiv) exponent algebra

Problem 6 (Chebyshev Bound) (20 points).
Albert has a gambling problem. He plays 240 hands of draw poker, 120 hands of black jack, and 40 hands of stud poker per day. He wins a hand of draw poker with probability 1/6, a hand of black jack with probability 1/2, and a hand of stud poker with probability 1/5. Let W be the the number of hands that Albert wins in a day.

(a) What is $\text{Ex}[W]$?

(b) What would the Markov bound be on the probability that Albert will win at least 216 hands on a given day?

(c) Assume that the outcomes of the card games are pairwise independent. Write a fraction equal to the variance of the number of hands won per day.

(d) What would the Chebyshev bound be on the probability that Albert will win at least 216 hands on a given day? Express your answer as a simple arithmetic expression.

Problem 7 (**Big/Little Oh**) (**16 points**).
Let $f(n) = n^3$. For each function $g(n)$ in the table below, write "yes" or "no" in each table entry to indicate which of the indicated asymptotic relations hold.

$g(n)$	$f = O(g)$	$f = o(g)$	$g = O(f)$	$g = o(f)$
$6 - 5n - 4n^2 + 3n^3$				
$n^3 \log n$				
$(\sin(\pi n/2) + 2)\, n^3$				
$n^{\sin(\pi n/2)+2}$				
$\log n!$				
$e^{0.2n} - 100n^3$				

Proof and Concept Questions

Problem 8 (Modular Inverse) (15 points).
Explain why 1059 does not have an inverse modulo 1412.

Problem 9 (Stable Marriage) (15 points).
In the Mating Ritual for stable marriages between an equal number of boys and girls, explain why there must be a girl to whom no boy proposes (serenades) until the last day.

Problem 10 (Well Ordering Principle) (25 points).
Use the Well Ordering Principle to prove that

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3} \tag{*}$$

for all integers $n \geq 1$.

Problem 11 (**Induction**) (**35 points**).
The *2-3-averaged numbers* are a subset, N23, of the real interval [0, 1] defined recursively as follows:

Base cases: $0, 1 \in \text{N23}$.

Constructor case: If a, b are in N23, then so is $L(a, b)$ where

$$L(a, b) ::= \frac{2a + 3b}{5}.$$

(a) Use ordinary induction or the Well-Ordering Principle to prove that

$$\left(\frac{3}{5}\right)^n \in \text{N23}$$

for all nonnegative integers n. **(Do not overlook part (b) on the next page.)**

(b) Prove by Structural Induction that the product of two 2-3-averaged numbers is also a 2-3-averaged number.

Hint: Prove by structural induction on c that, if $d \in$ N23, then $cd \in$ N23.