Final Examination

Your name: ______________________________

- This exam is **closed book** except for two 2-sided cribsheets. Total time is 180 minutes.
- Write your solutions in the space provided. If you need more space, **write on the back** of the sheet containing the problem.
- In answering the following questions, you may use without proof any of the results from class or text.

DO NOT WRITE BELOW THIS LINE

Problem	Points	Grade	Grader
1	12		
2	15		
3	12		
4	12		
5	10		
6	12		
7	20		
8	20		
9	20		
10	20		
11	12		
12	15		
13	20		
Total	200		

Short-Answer Questions

The following questions are short-answer. The graders will not read explanations, so do not spend time including them.

Problem 1 (Quantifiers, Law of Large Numbers) (12 points).
Let $G_1, G_2, G_3, \ldots,$ be an infinite sequence of pairwise independent random variables with the same expectation μ and the same finite variance. Let

$$f(n, \epsilon) ::= \Pr\left[\left| \frac{\sum_{i=1}^{n} G_i}{n} - \mu \right| \leq \epsilon \right].$$

The Weak Law of Large Numbers can be expressed as a logical formula of the form:

$$\forall \epsilon > 0 \;\; \mathbf{Q}_1 \;\; \mathbf{Q}_2 \;\; \mathbf{Q}_3 .\; f(n, \epsilon) \geq 1 - \delta$$

where $\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3$ is a sequence of three quantifiers from among:

$$\begin{array}{cccc} \forall n, & \exists n, & \forall n \geq n_0, & \exists n \geq n_0. \\ \forall n_0, & \exists n_0, & \forall n_0 \geq n, & \exists n_0 \geq n. \\ \forall \delta, & \exists \delta, & \forall \delta > 0, & \exists \delta > 0. \end{array}$$

Here the n, n_0 range over nonnegative integers, and δ, ϵ range over nonnegative real numbers.

Write out the proper sequence $\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3$.

Problem 2 (Counting Relations) (15 points).
This problem is about binary relations on the set of integers in the interval $[1..n]$ and graphs whose vertex set is $[1..n]$.

(a) How many simple graphs are there?

(b) How many asymmetric binary relations are there?

(c) How many linear strict partial orders are there?

Problem 3 (Remainder Arithmetic) (12 points).
You should not need to do any calculation to answer the next two questions:

(a) What is rem(3^{99}, 97)?

(b) What is rem($96^{123456789}$, 97)?

Hint: $n - 1 \equiv -1 \pmod{n}$.

Problem 4 (Scheduling) (12 points).
Let B^4 be the length-4 binary vectors partially ordered *coordinatewise*, that is,

$$b_1 b_2 b_3 b_4 \leq c_1 c_2 c_3 c_4$$

iff $b_i \leq c_i$ for $i = 1, 2, 3, 4$. For example,

$$\texttt{1001} \leq \texttt{1011} \leq \texttt{1111}$$

`1001` incomparable to `0101`.

(a) Give an example of a maximum length chain for in B^4.

(b) Give an example of an antchain of size 6 in B^4.

(c) Suppose the partial order on B^4 describes scheduling constraints on 16 unit-time tasks. What is the length of a minimum time **3**-processor schedule for B^4?

Problem 5 (Stable Marriage) (10 points).
We are interested in invariants of the Mating Ritual for finding stable marriages. Let Angelina and Jen be two of the girls, and Keith and Tom be two of the boys.

Which of the following predicates are invariants of the Mating Ritual no matter what the preferences are among the boys and girls?

(a) Tom is serenading Jen.

(b) Tom is not serenading Jen.

(c) Tom's list of girls to serenade is empty.

(d) Jen is crossed off Keith's list and Keith prefers Jen to anyone he is serenading.

(e) Jen is the only girl on Keith's list.

Proof and Concept Questions

Problem 6 (**GCD**) (**12 points**).

Let a, b be positive integers. Suppose no integer linear combination of a and b equals two. Explain why no integer linear combination of a^2 and b^2 equals eight.

Hint: Rephrase in terms of gcd.

Problem 7 (Well Ordering Principle) (20 points).
Prove by the *Well Ordering Principle*[1] that for all positive integers, n:

$$\sum_{i=1}^{n}(2i-1) = n^2.$$

[1] Proofs by other methods such as induction or by appeal to known formulas for similar sums will not receive credit.

Problem 8 (Induction, Simple Graphs) (20 points).
If a simple graph has e edges, v vertices, and k connected components, then it has at least $e - v + k$ cycles.

Prove this by induction on the number of edges e. You should begin by carefully stating an induction hypothesis $P(e)$.

Problem 9 (Structural Induction) (20 points).
The set OBT of *Ordered Binary Trees* is defined recursively as follows:

Base case: $\langle\textbf{leaf}\rangle$ is an OBT, and

Constructor case: if R and S are OBT's, then $\langle\textbf{node}, R, S\rangle$ is an OBT.

If T is an OBT, let n_T be the number of **node** labels in T and l_T be the number of **leaf** labels in T. Prove by structural induction that for all $T \in$ OBT,

$$l_T = n_T + 1.$$

Problem 10 (Diagonalization/Jections) (20 points).
A subset of the nonnegative integers $\mathbb{N}$ is called *lonely* when it doesn't contain any pair of consecutive integers. For example, the set $\{1, 5, 25, 125, \ldots\}$ of powers of 5 is lonely, but the set of primes $\{2, 3, 5, 7, 11, \ldots\}$ is not lonely because it contains both 2 and 3.

Let L be the set of lonely subsets of $\mathbb{N}$. Show that L is uncountable.

Hint: Instead of the standard diagonal with slope -1, try one with slope -1/2; alternatively, describe a inj or surj relation with pow($\mathbb{N}$).

Problem 11 (Big Oh) (12 points).
Let $f, g : \mathbb{N} \to \mathbb{N}$ be defined as

$$f(n) ::= n^n, \qquad g(n) ::= \begin{cases} n^{n-(1/2)} & \text{if } n \text{ is odd,} \\ n^{n+(1/2)} & \text{if } n \text{ is even.} \end{cases}$$

(a) Verify that $f \neq O(g)$.

(b) Verify that $g \neq O(f)$.

Problem 12 (**Variance**) (**15 points**).

You are playing a game where you get n turns. Each of your turns involves flipping a coin a number of times. On the first turn, you have 1 flip, on the second turn you have two flips, and so on until your nth turn when you flip the coin n times. All the flips are mutually independent.

The coin you are using is biased to flip Heads with probability p. You *win* a turn if you flip all Heads. Let W be the number of winning turns.

(a) Write a closed-form (no summations) expression for $\text{Ex}[W]$.

(b) Write a closed-form expression for $\text{Var}[W]$.

Problem 13 (Chebyshev Bound) (20 points).
You have a biased coin which flips Heads with probability p. You flip the coin n times. The coin flips are all mutually independent. Let H be the number of Heads.

(a) Write a closed-form (no summations) expression in terms of p and n for $\text{Ex}[H]$, the expected number of Heads.

(b) Write a closed-form expression in terms of p and n for $\text{Var}[H]$, the variance of the number of Heads.

(c) Write a closed-form expression in terms of p for the upper bound that Markov's Theorem gives for the probability that the number of Heads is larger than the expected number by at least 1% of the number of flips, that is, by $n/100$.

(d) Show that the upper bound given by Chebyshev's Theorem for the probability that H differs from $\text{Ex}[H]$ by at least $n/100$ is

$$100^2 \frac{p(1-p)}{n}.$$