Final Examination, Unit 2

Your name:____

- This exam is **closed book** except for two 2-sided cribsheets. Total time is 180 minutes.
- Write your solutions in the space provided. If you need more space, **write on the back** of the sheet containing the problem.
- In answering the following questions, you may use without proof any of the results from class or text.

DO NOT WRITE BELOW THIS LINE

Problem	Points	Grade	Grader
1	30		
2	20		
Total	50		

Problem 1 (State Machines, Induction) (30 points).

Token replacing-1-3 is a single player game using a set of tokens, each colored black or white. In each move, a player can replace a black token with three white tokens, or replace a white token with three black tokens. We can model this game as a state machine whose states are pairs (b, w) of nonnegative integers, where b is the number of black tokens and w the number of white ones.

The game has two possible start states: $(5, 4)$ or $(4, 3)$. We call a state *reachable* if it is reachable from *at least one* of the two start states.

We call a state (b, w) *eligible* when

$$\text{rem}(b - w,\ 4) = 1, \text{ AND} \tag{1}$$

$$\min\{b, w\} \geq 3. \tag{2}$$

(Recall that rem$(n,\ 4)$ denotes the number $0 \leq r \leq 3$ such that $n = 4q + r$ for some $q \in \mathbb{Z}$.)

This problem examines the connection between eligible states and reachable states.

(a) Give an example of a reachable state that is not eligible. Explain.

(b) Show that the derived variable $b + w$ is strictly increasing. Conclude that state $(3, 2)$ is not reachable.

(c) Verify that rem$(3b + w,\ 8)$ is a derived variable that is constant. Conclude that no state is reachable from both start states.

(d) Suppose (b, w) is eligible and $b \geq 6$. Verify that $(b - 3, w + 1)$ is eligible.

We now prove that every eligible state is reachable. For the rest of the problem, you may—and should—**assume** the following Fact:

Fact. If $\max\{b, w\} \leq 5$ and (b, w) is eligible, then (b, w) is reachable.

(This is easy to verify since there are only nine states with $b, w \in \{3, 4, 5\}$, but don't waste time doing this.)

(e) Define the predicate $P(n)$ to be:

$$\forall (b, w).[b + w = n \text{ AND } (b, w) \text{ is eligible}] \text{ IMPLIES } (b, w) \text{ is reachable.}$$

Prove **carefully by strong induction** that $P(n)$ is true for every integer $n \geq 0$. *Hint:* Prove and use the fact that $P(n - 1)$ IMPLIES $P(n + 1)$.

Problem 2 (**Relations, Predicates**) (**20 points**). **(a)** Let R be a binary relation on a set D. Each of the following formulas expresses the fact that R has a familiar relational property such as reflexivity, asymmetry, transitivity, etc. For each of the five predicate formulas below, identify the **name** of that property.

(1) $\forall c, d. \quad c\ R\ d$ IFF $d\ R\ c$ ____________

(2) $\forall d. \quad$ NOT$(d\ R\ d)$ ____________

(3) $\forall c, d. \quad c\ R\ d$ IMPLIES NOT$(d\ R\ c)$ ____________

(4) $\forall b, d. \quad [\exists c. (b\ R\ c$ AND $c\ R\ d)]$ IMPLIES $b\ R\ d$ ____________

(5) $\forall c \neq d. \quad c\ R\ d$ IMPLIES NOT$(d\ R\ c)$ ____________

(b) Below are five relational formulas encoding the **same** five familiar properties as in part (a), in scrambled order. Match these five relational formulas to the predicate formulas from part (a) by writing the **number** that each corresponds to. Your answers should simply be the numbers 1–5 in some order.

(6) $R \cap \mathrm{Id}_D = \emptyset$ ______

(7) $R \subseteq R^{-1}$ ______

(8) $R \circ R \subseteq R$ ______

(9) $R \cap R^{-1} \subseteq \mathrm{Id}_D$ ______

(10) $R \cap R^{-1} = \emptyset$ ______

In these formulas, Id_D is the "identity relation" on D, defined by $\mathrm{Id}_D ::= \{(d, d) \mid d \in D\}$.