In-Class Problems Week 9, Mon.

Problem 1.
Which of the items below are simple-graph properties preserved under isomorphism?

(a) There is a cycle that includes all the vertices.

(b) The vertices are numbered 1 through 7.

(c) The vertices can be numbered 1 through 7.

(d) There are two degree 8 vertices.

(e) Two edges are of equal length.

(f) No matter which edge is removed, there is a path between any two vertices.

(g) There are two cycles that do not share any vertices.

(h) The vertices are sets.

(i) The graph can be drawn in a way that all the edges have the same length.

(j) No two edges cross.

(k) The OR of two properties that are preserved under isomorphism.

(l) The negation of a property that is preserved under isomorphism.

Problem 2.
For each of the following pairs of simple graphs, either define an isomorphism between them, or prove that there is none. (We write ab as shorthand for $\langle a—b \rangle$.)

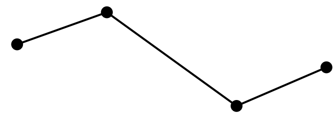
(a)

$$G_1 \text{ with } V_1 = \{1,2,3,4,5,6\},\ E_1 = \{12,23,34,14,15,35,45\}$$
$$G_2 \text{ with } V_2 = \{1,2,3,4,5,6\},\ E_2 = \{12,23,34,45,51,24,25\}$$

(b)

$$G_3 \text{ with } V_3 = \{1,2,3,4,5,6\},\ E_3 = \{12,23,34,14,45,56,26\}$$
$$G_4 \text{ with } V_4 = \{a,b,c,d,e,f\},\ E_4 = \{ab,bc,cd,de,ae,ef,cf\}$$

Problem 3.
Let's say that a graph has "two ends" if it has exactly two vertices of degree 1 and all its other vertices have degree 2. For example, here is one such graph:



(a) A *line graph* is a graph whose vertices can be listed in a sequence with edges between consecutive vertices only. So the two-ended graph above is also a line graph of length 4.

Prove that the following theorem is false by drawing a counterexample.

False Theorem. *Every two-ended graph is a line graph.*

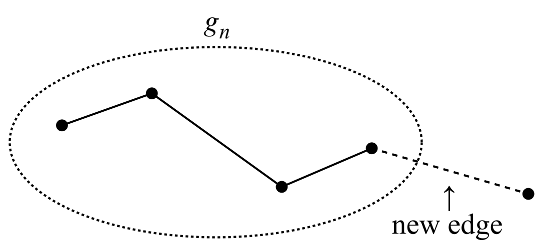
(b) Point out the first erroneous statement in the following bogus proof of the false theorem and describe the error.

Bogus proof. We use induction. The induction hypothesis is that every two-ended graph with n edges is a line graph.

Base case ($n = 1$): The only two-ended graph with a single edge consists of two vertices joined by an edge:

Sure enough, this is a line graph.

Inductive case: We assume that the induction hypothesis holds for some $n \geq 1$ and prove that it holds for $n + 1$. Let G_n be any two-ended graph with n edges. By the induction assumption, G_n is a line graph. Now suppose that we create a two-ended graph G_{n+1} by adding one more edge to G_n. This can be done in only one way: the new edge must join one of the two endpoints of G_n to a new vertex; otherwise, G_{n+1} would not be two-ended.



Clearly, G_{n+1} is also a line graph. Therefore, the induction hypothesis holds for all graphs with $n + 1$ edges, which completes the proof by induction.

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Problem 4.
The average degree of the vertices in an n-vertex graph is twice the average number of edges per vertex. Explain why.

Problem 5.
A researcher analyzing data on heterosexual sexual behavior in a group of m males and f females found that within the group, the male average number of female partners was 10% larger that the female average number of male partners.

(a) Comment on the following claim. "Since we're assuming that each encounter involves one man and one woman, the average numbers should be the same, so the males must be exaggerating."

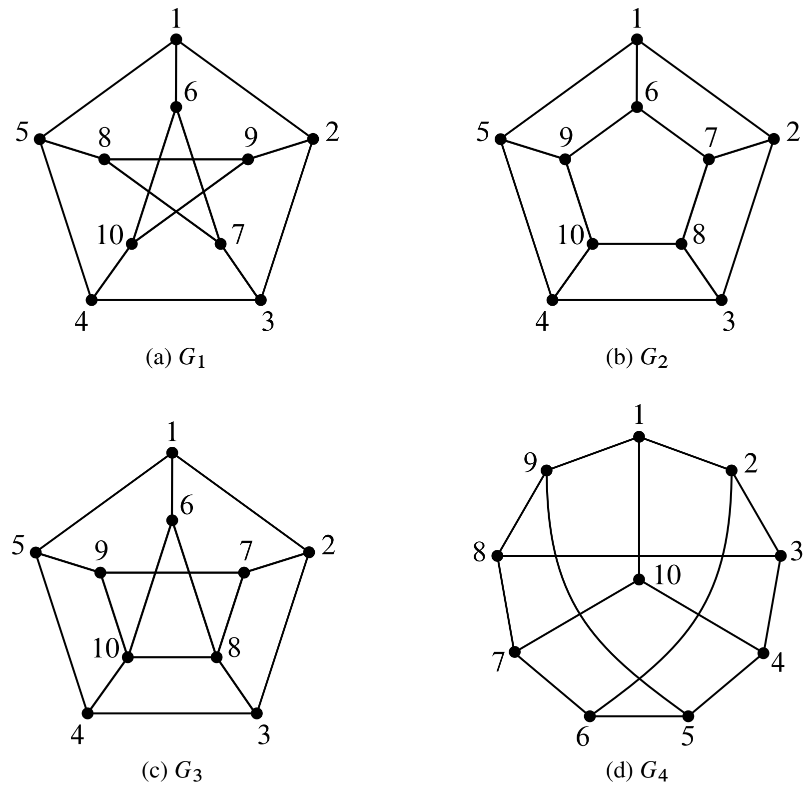

Figure 1 Which graphs are isomorphic?

(b) For what constant c is $m = c \cdot f$?

(c) The data shows that approximately 20% of the females were virgins, while only 5% of the males were. The researcher wonders how excluding virgins from the population would change the averages. If he knew graph theory, the researcher would realize that the nonvirgin male average number of partners will be $x(f/m)$ times the nonvirgin female average number of partners. What is x?

(d) For purposes of further research, it would be helpful to pair each female in the group with a unique male in the group. Explain why this is not possible.

Problem 6.
Determine which among the four graphs pictured in Figure 1 are isomorphic. For each pair of isomorphic graphs, describe an isomorphism between them. For each pair of graphs that are not isomorphic, give a property that is preserved under isomorphism such that one graph has the property, but the other does not. For at least one of the properties you choose, *prove* that it is indeed preserved under isomorphism (you only need prove one of them).