In-Class Problems Week 7, Fri.

Problem 1. (a) Indicate which of the following relations below are equivalence relations, (**Eq**), strict partial orders (**SPO**), weak partial orders (**WPO**). For the partial orders, also indicate whether it is *linear* (**Lin**).

If a relation is none of the above, indicate whether it is *transitive* (**Tr**), *symmetric* (**Sym**), or *asymmetric* (**Asym**).

(i) The relation $a = b + 1$ between integers a, b,

(ii) The superset relation $\supseteq$ on the power set of the integers.

(iii) The empty relation on the set of rationals.

(iv) The divides relation on the nonegative integers $\mathbb{N}$.

(v) The divides relation on all the integers $\mathbb{Z}$.

(vi) The divides relation on the positive powers of 4.

(vii) The relatively prime relation on the nonnegative integers.

(viii) The relation "has the same prime factors" on the integers.

(b) A set of functions $f, g : D \to \mathbb{R}$ can be partially ordered by the $\leq$ relation, where

$$[f \leq g] ::= \forall d \in D.\ f(d) \leq g(d).$$

Let L be the set of functions $f : \mathbb{R} \to \mathbb{R}$ of the form

$$f(x) = ax + b$$

for constants $a, b \in \mathbb{R}$.

Describe an infinite chain and an infinite anti-chain in L.

Hint: Think about parallel lines.

Problem 2.
Let R_1 and R_2 be two equivalence relations on a set A. Prove or give a counterexample to the claims that the following are also equivalence relations:

(a) $R_1 \cap R_2$.

(b) $R_1 \cup R_2$.

Problem 3.
Let S be a sequence of n different numbers. A *subsequence* of S is a sequence that can be obtained by deleting elements of S.

For example, if S is

$$(6, 4, 7, 9, 1, 2, 5, 3, 8),$$

then 647 and 7253 are both subsequences of S (for readability, we have dropped the parentheses and commas in sequences, so 647 abbreviates $(6, 4, 7)$, for example).

An *increasing subsequence* of S is a subsequence of whose successive elements get larger. For example, 1238 is an increasing subsequence of S. Decreasing subsequences are defined similarly; 641 is a decreasing subsequence of S.

(a) List all the maximum-length increasing subsequences of S, and all the maximum-length decreasing subsequences.

Now let A be the *set* of numbers in S. (So A is the integers $[1..9]$ for the example above.) There are two straightforward linear orders for A. The first is numerical order where A is ordered by the $<$ relation. The second is to order the elements by which comes first in S; call this order $<_S$. So for the example above, we would have

$$6 <_S 4 <_S 7 <_S 9 <_S 1 <_S 2 <_S 5 <_S 3 <_S 8$$

Let $\prec$ be the product relation of the linear orders $<_s$ and $<$. That is, $\prec$ is defined by the rule

$$a \prec a' \quad ::= \quad a < a' \text{ AND } a <_S a'.$$

So $\prec$ is a partial order on A (Section 10.9).

(b) Draw a diagram of the partial order $\prec$ on A. What are the maximal and minimal elements?

(c) Explain the connection between increasing and decreasing subsequences of S, and chains and anti-chains under $\prec$.

(d) Prove that every sequence S of length n has an increasing subsequence of length greater than $\sqrt{n}$ or a decreasing subsequence of length at least $\sqrt{n}$.

Supplemental problem:

Problem 4.
For any total function $f : A \to B$ define a relation $\equiv_f$ by the rule:

$$a \equiv_f a' \quad \text{iff} \quad f(a) = f(a'). \tag{1}$$

(a) Sketch a proof that $\equiv_f$ is an equivalence relation on A.

(b) Prove that every equivalence relation R on a set A is equal to $\equiv_f$ for the function $f : A \to \text{pow}(A)$ defined as

$$f(a) ::= \{a' \in A \mid a \; R \; a'\}.$$

That is, $f(a) = R(a)$.