In-Class Problems Week 6, Fri.

Problem 1.
Let P be any set predicate, and define $\mathcal{S}_P$ as the collection of all sets satisfying P:

$$\mathcal{S}_P ::= \{S \in \text{Sets} \mid P(S)\}.$$

(a) Give an example P such that $\mathcal{S}_P$ is not a set.

(b) Give an example P such that $\mathcal{S}_P$ is a set.

(c) Prove that if $\mathcal{S}_P$ is a set, then $P(\mathcal{S}_P)$ must be false.

Problem 2.
Let $R : A \to A$ be a binary relation on a set A. If $a_1 \; R \; a_0$, we'll say that a_1 is "R-smaller" than a_0. R is called *well founded* when there is no infinite "R-decreasing" sequence:

$$\cdots R \; a_n \; R \cdots R \; a_1 \; R \; a_0, \tag{1}$$

of elements $a_i \in A$.

For example, if $A = \mathbb{N}$ and R is the $<$-relation, then R is well founded because if you keep counting down with nonnegative integers, you eventually get stuck at zero:

$$0 < \cdots < n - 1 < n.$$

But you can keep counting up forever, so the $>$-relation is not well founded:

$$\cdots > n > \cdots > 1 > 0.$$

Also, the $\leq$-relation on $\mathbb{N}$ is not well founded because a *constant* sequence of, say, 2's, gets $\leq$-smaller forever:

$$\cdots \leq 2 \leq \cdots \leq 2 \leq 2.$$

(a) If B is a subset of A, an element $b \in B$ is defined to be *R-minimal in B* iff there is no R-smaller element in B. Prove that $R : A \to A$ is well founded iff every nonempty subset of A has an R-minimal element.

A logic *formula of set theory* has only predicates of the form "$x \in y$" for variables x, y ranging over sets, along with quantifiers and propositional operations. For example,

$$\text{isempty}(x) ::= \forall w. \; \text{NOT}(w \in x)$$

is a formula of set theory that means that "x is empty."

(b) Write a formula member-minimal(u, v) of set theory that means that u is $\in$-minimal in v.

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(c) The Foundation axiom of set theory says that $\in$ is a well founded relation on sets. Express the Foundation axiom as a formula of set theory.

You may use "member-minimal" and "isempty" in your formula as abbreviations for the formulas defined above.

(d) Explain why the Foundation axiom implies that no set is a member of itself.

Problem 3.
Cantor's Theorem implies there is no bijection $f : \text{pow}(A) \to A$. Without appeal to Cantor's Theorem, prove this by contradiction. In particular, assume there was such an f, and define

$$W_f ::= \{a \in A \mid a \notin f^{-1}(a)\}.$$

Show that

$$f(W_f) \in f^{-1}(f(W_f) \quad \text{IFF} \quad f(W_f) \notin f^{-1}(f(W_f)). \tag{*}$$

Problem 4.
In this problem, structural induction provides a simple proof about some utterly infinite objects.

Every pure set defines a two-person game in which a player's move consists of choosing any element of the game. The two players alternate moves, and a player loses when it is their turn to move and there is no move to make. That is, whoever moves to the empty set is a winner, because the next player has no move.

So we think of a set R as the initial "board position" of a *set game*. The player who goes first in R is called the *Next* player, and the player who moves second in R is called the *Previous* player. When the Next player moves to an $S \in R$, the game continues with the new set game S in which the Previous player moves first.

Prove by structural induction on the Definition 8.3.1 of recursive sets Recset that for every set game, either the Previous player or the Next player has a winning strategy.[1] Reminder:

Definition. The class of *recursive sets* Recset is defined as follows:

Base case: The empty set $\emptyset$ is a Recset.

Constructor step: If S is a nonempty set of Recset's, then S is a Recset.

By Theorem 8.3.2, every set is recursive.

[1] Set games are called "uniform" because the two players have the *same* objective: to leave the other player stuck with no move to make. In more general games, the two players have different objectives, for example, one wants to maximize the final payoff and the other wants to minimize it (Problem 7.35).