In-Class Problems Week 5, Wed.

Problem 1.
The Elementary 18.01 Functions (F18's) are the set of functions of one real variable defined recursively as follows:

Base cases:

- The identity function $\mathrm{id}(x) ::= x$ is an F18,
- any constant function is an F18,
- the sine function is an F18,

Constructor cases:
If f, g are F18's, then so are

1. $f + g$, fg, 2^g,
2. the inverse function f^{-1},
3. the composition $f \circ g$.

(a) Prove that the function $1/x$ is an F18.

Warning: Don't confuse $1/x = x^{-1}$ with the inverse id^{-1} of the identity function $\mathrm{id}(x)$. The inverse id^{-1} is equal to id.

(b) Prove by Structural Induction on this definition that the Elementary 18.01 Functions are *closed under taking derivatives*. That is, show that if $f(x)$ is an F18, then so is $f' ::= df/dx$. (Just work out 2 or 3 of the most interesting constructor cases; you may skip the less interesting ones.)

Problem 2.
Here is a simple recursive definition of the set E of even integers:

Definition. Base case: $0 \in E$.
Constructor cases: If $n \in E$, then so are $n + 2$ and $-n$.

Provide similar simple recursive definitions of the following sets:

(a) The set $S ::= \{2^k 3^m 5^n \in \mathbb{N} \mid k, m, n \in \mathbb{N}\}$.

(b) The set $T ::= \{2^k 3^{2k+m} 5^{m+n} \in \mathbb{N} \mid k, m, n \in \mathbb{N}\}$.

(c) The set $L ::= \{(a, b) \in \mathbb{Z}^2 \mid (a - b) \text{ is a multiple of } 3\}$.

Let L' be the set defined by the recursive definition you gave for L in the previous part. Now if you did it right, then $L' = L$, but maybe you made a mistake. So let's check that you got the definition right.

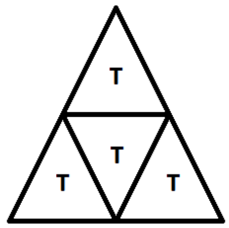

Figure 1 DET T' from Four Copies of DET T

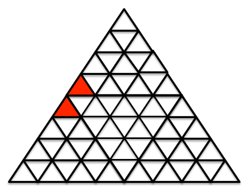

Figure 2 Subtriangles and "middle" subtriangles

(d) Prove by structural induction on your definition of L' that

$$L' \subseteq L.$$

(e) Confirm that you got the definition right by proving that

$$L \subseteq L'.$$

(f) (*Optional*) See if you can give an *unambiguous* recursive definition of L.

Problem 3.
Divided Equilateral Triangles (DETs) were defined in an earlier class problem 5.10 as follows:

- **Base case:** An equilateral triangle with unit length sides is a DET whose only subtriangle is itself.
- If $T ::=$  is a DET, then the equilateral triangle T' built out of four copies of T as shown in Figure 1 is also a DET, and the subtriangles of T' are exactly the subtriangles of each of the copies of T.

Note that "subtriangles" correspond to *unit* subtriangles as illustrated in Figure 2. The four large triangles from which a new triangle is made are *not* subtriangles (unless the side length is two).

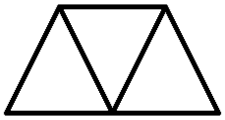

Figure 3 Trapezoid from Three Triangles

Properties of DETs were proved earlier by induction on the length of a side of the triangle. Recognizing that the definition of DETs is recursive, we can instead prove properties of DETs by structural induction.

(a) Prove by structural induction that a DET with one of its corner subtriangles removed can be tiled with trapezoids built out of three subtriangles as in Figure 3.

(b) Explain why a DET with a middle triangle removed from one side can also be tiled by trapezoids. (If the side length is even, there are two "middle" subtriangles on each side. The middle subtriangles on the left hand edge are colored red in Figure 2.)

(c) In tiling a large square using L-shaped blocks as described in the text, there was a tiling with any single subsquare removed. Part (b) indicates that trapezoid-tilings are possible for DETs with a non-corner subtriangle removed, so it's natural to make the mistaken guess that DETs have a corresponding property:

False Claim. *A DET with any single subtriangle removed can be trapezoid-tiled.*

We can try to prove the claim by structural induction as in part (a).

Bogus proof. The claim holds vacuously in the base case of a DET with a single subtriangle.

Now let T' be a DET made of four copies of a DET T, and suppose we remove an arbitrary subtriangle from T'.

The removed subtriangle must be a subtriangle of one of the copies of T. The copies are the same, so for definiteness we assume the subtriangle was removed from copy 1. Then by structural induction hypothesis, copy 1 can be trapezoid-tiled, and then the other three copies of T can be trapezoid-tiled exactly as in the solution to part(a). This yields a complete trapezoid-tiling of T' with the arbitrary subtriangle removed.

We conclude by structural induction that any DET with any subtriangle removed can be trapezoid-tiled. ■

What's wrong with the proof?

Hint: Find a counter-example, and show where the proof breaks down.

(We don't know if there is a simple characterization of exactly which subtriangles can be removed to allow a trapezoid tiling.)

Problem 4.

We're going to characterize a large category of games as a recursive data type and then prove, by structural induction, a fundamental theorem about game strategies. We are interested in two person games of perfect information that end with a numerical score. Chess and Checkers would count as value games using the values $1, -1, 0$ for a win, loss or draw for the first player. The game of Go really does end with a *score* based on the number of white and black stones that remain at the end.

Here's the formal definition:

Definition. Let V be a nonempty set of real numbers. The class VG of *V-valued two-person deterministic games of perfect information* is defined recursively as follows:

Base case: A value $v \in V$ is a VG known as a *payoff*.

Constructor case: If G is a nonempty set of VG's, then G is a VG. Each game $M \in G$ is called a possible *first move* of G.

A *strategy* for a player is a rule that tells the player which move to make whenever it is their turn. That is, a strategy is a function s from games to games with the property that $s(G) \in G$ for all games G. Given which player has the first move, a pair of strategies for the two players determines exactly which moves the players will choose. So the strategies determine a unique play of the game and a unique payoff.[1]

The *max-player* wants a strategy that guarantees as high a payoff as possible, and the *min-player* wants a strategy that guarantees as low a payoff as possible.

The Fundamental Theorem for deterministic games of perfect information says that in any game, each player has an optimal strategy, and these strategies lead to the same payoff. More precisely,

Theorem (Fundamental Theorem for VG's)**.** *Let V be a finite set of real numbers and G be a V-valued VG. Then there is a value $v \in V$, called a* max-value max_G *for G, such that if the* max-player moves first,

- *the max-player has a strategy that will finish with a payoff of* at least max_G*, no matter what strategy the min-player uses, and*

- *the min-player has a strategy that will finish with a payoff of* at most max_G*, no matter what strategy the max-player uses.*

It's worth a moment for the reader to observe that the definition of $\max_G$ implies that if there is one for G, it is unique. So if the max-player has the first move, the Fundamental Theorem means that there's no point in playing the game: the min-player may just as well pay the max-value to the max-player.

(a) Prove the Fundamental Theorem for VG's.

Hint: VG's are a recursively defined data type, so the basic method for proving that all VG's have some property is structural induction on the definition of VG. Since the min-player moves first in whichever game the max-player picks for their first move, the induction hypothesis will need to cover that case as well.

(b) (OPTIONAL). State some reasonable generalization of the Fundamental Theorem to games with an infinite set V of possible payoffs.

[1] We take for granted the fact that no VG has an infinite play. The proof of this by structural induction is essentially the same as that for win-lose games given in the text.