In-Class Problems Week 4, Mon.

Problem 1.

The *inverse* R^{-1} of a binary relation R from A to B is the relation from B to A defined by:

$$b \; R^{-1} \; a \quad \text{iff} \quad a \; R \; b.$$

In other words, you get the diagram for R^{-1} from R by "reversing the arrows" in the diagram describing R. Now many of the relational properties of R correspond to different properties of R^{-1}. For example, R is *total* iff R^{-1} is a *surjection*.

Fill in the remaining entries is this table:

R is	iff R^{-1} is
total	a surjection
a function	
a surjection	
an injection	
a bijection	

Hint: Explain what's going on in terms of "arrows" from A to B in the diagram for R.

Arrow Properties

Definition. A binary relation, R is

- is a *function* when it has the [≤ 1 arrow **out**] property.
- is *surjective* when it has the [≥ 1 arrows **in**] property. That is, every point in the right-hand, codomain column has at least one arrow pointing to it.
- is *total* when it has the [≥ 1 arrows **out**] property.
- is *injective* when it has the [≤ 1 arrow **in**] property.
- is *bijective* when it has both the [$= 1$ arrow **out**] and the [$= 1$ arrow **in**] property.

Problem 2.

Assume $f : A \to B$ is total function, and A is finite. Replace the $\star$ with one of $\leq, =, \geq$ to produce the *strongest* correct version of the following statements:

(a) $|f(A)| \star |B|$.

(b) If f is a surjection, then $|A| \star |B|$.

(c) If f is a surjection, then $|f(A)| \star |B|$.

(d) If f is an injection, then $|f(A)| \star |A|$.

(e) If f is a bijection, then $|A| \star |B|$.

Problem 3.
Let $R : A \to B$ be a binary relation. Use an arrow counting argument to prove the following generalization of the Mapping Rule 1.

Lemma. *If R is a function, and $X \subseteq A$, then*

$$|X| \geq |R(X)|.$$

Problem 4.
Let $A = \{a_0, a_1, \ldots, a_{n-1}\}$ be a set of size n, and $B = \{b_0, b_1, \ldots, b_{m-1}\}$ a set of size m. Prove that $|A \times B| = mn$ by defining a simple bijection from $A \times B$ to the nonnegative integers from 0 to $mn - 1$.

Problem 5. **(a)** Prove that if A surj B and B surj C, then A surj C.

(b) Explain why A surj B iff B inj A.

(c) Conclude from (a) and (b) that if A inj B and B inj C, then A inj C.

(d) According to the official definition, A inj B requires a total injective *relation* ([$\geq$ 1 out, $\leq$ 1 in]). Explain why A inj B iff there is a total injective *function* ([= 1 out, $\leq$ 1 in]) from A to B.