In-Class Problems Week 3, Wed.

Problem 1.
Say a number of cents is *makeable* if it is the value of some set of 6 cent and 15 cent stamps. Use the Well Ordering Principle to show that every integer that is a multiple of 3 and greater than or equal to twelve is makeable.

Problem 2.
Use the *Well Ordering Principle* [1] to prove that

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}. \tag{1}$$

for all nonnegative integers n.

Problem 3.
Euler's Conjecture in 1769 was that there are no positive integer solutions to the equation

$$a^4 + b^4 + c^4 = d^4.$$

Integer values for a, b, c, d that do satisfy this equation were first discovered in 1986. So Euler guessed wrong, but it took more than two centuries to demonstrate his mistake.

Now let's consider Lehman's equation, similar to Euler's but with some coefficients:

$$8a^4 + 4b^4 + 2c^4 = d^4 \tag{2}$$

Prove that Lehman's equation (2) really does not have any positive integer solutions.

Hint: Consider the minimum value of a among all possible solutions to (2).

Problem 4.
An n-bit AND-circuit has 0-1 valued inputs $a_0, a_1, \ldots, a_{n-1}$ and one output c whose value will be

$$c = a_0 \text{ AND } a_1 \text{ AND } \cdots \text{ AND } a_{n-1}.$$

There are various ways to design an n-bit AND-circuit. A *serial* design is simply a series of AND-gates, each with one input being a circuit input a_i and the other input being the output of the previous gate as shown in Figure 1.

We can also use a *tree* design. A 1-bit tree design is just a wire, that is $c ::= a_1$. Assuming for simplicity that n is a power of two, an n-input tree circuit for $n > 1$ simply consists of two $n/2$-input tree circuits whose outputs are AND'd to produce output c, as in Figure 2. For example, a 4-bit tree design circut is shown in Figure 3.

[1] Proofs by other methods such as induction or by appeal to known formulas for similar sums will not receive full credit.

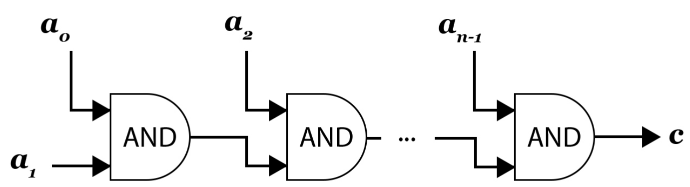

Figure 1 A serial AND-circuit.

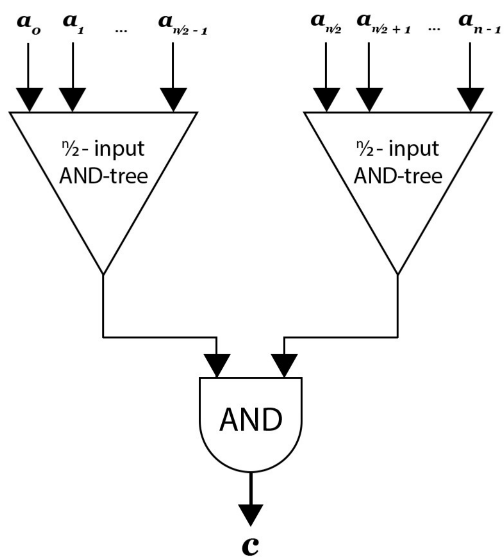

Figure 2 An n-bit AND-tree circuit.

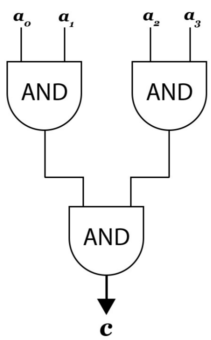

Figure 3 A 4-bit AND-tree circuit.

(a) How many AND-gates are in the n-input serial circuit?

(b) The "speed" or *latency* of a circuit is the largest number of gates on any path from an input to an output. Briefly explain why the tree circuit is *exponentially faster* than the serial circuit.

(c) Assume n is a power of two. Prove using the Well Ordering Principle that the n-input tree circuit has $n - 1$ AND-gates.

Problem 5.

You are given a series of envelopes, respectively containing $1, 2, 4, \ldots, 2^m$ dollars. Define

> **Property** m: For any nonnegative integer less than 2^{m+1}, there is a selection of envelopes whose contents add up to *exactly* that number of dollars.

Use the Well Ordering Principle (WOP) to prove that Property m holds for all nonnegative integers m.

Hint: Consider two cases: first, when the target number of dollars is less than 2^m and second, when the target is at least 2^m.