In-Class Problems Week 3, Fri.

Problem 1.
Set Formulas and Propositional Formulas.
(a) Verify that the propositional formula $(P \text{ AND } \overline{Q}) \text{ OR } (P \text{ AND } Q)$ is equivalent to P.

(b) Prove that[1]

$$A = (A - B) \cup (A \cap B)$$

for all sets, A, B, by showing

$$x \in A \text{ IFF } x \in (A - B) \cup (A \cap B)$$

for all elements x using the equivalence of part (a) in a chain of IFF's.

Problem 2.
Subset take-away[2] is a two player game played with a finite set A of numbers. Players alternately choose nonempty subsets of A with the conditions that a player may not choose

- the whole set A, or
- any set containing a set that was named earlier.

The first player who is unable to move loses the game.

For example, if the size of A is one, then there are no legal moves and the second player wins. If A has exactly two elements, then the only legal moves are the two one-element subsets of A. Each is a good reply to the other, and so once again the second player wins.

The first interesting case is when A has three elements. This time, if the first player picks a subset with one element, the second player picks the subset with the other two elements. If the first player picks a subset with two elements, the second player picks the subset whose sole member is the third element. In both cases, these moves lead to a situation that is the same as the start of a game on a set with two elements, and thus leads to a win for the second player.

Verify that when A has four elements, the second player still has a winning strategy.[3]

Problem 3.
Forming a pair (a, b) of items a and b is a mathematical operation that we can safely take for granted. But when we're trying to show how all of mathematics can be reduced to set theory, we need a way to represent the pair (a, b) as a set.

[1] The *set difference* $A - B$ of sets A and B is

$$A - B ::= \{a \in A \mid a \notin B\}.$$

[2] From Christenson & Tilford, *David Gale's Subset Takeaway Game, American Mathematical Monthly, Oct. 1997*

[3] David Gale worked out some of the properties of this game and conjectured that the second player wins the game for any set A. This remains an open problem.

(a) Explain why representing (a, b) by $\{a, b\}$ won't work.

(b) Explain why representing (a, b) by $\{a, \{b\}\}$ won't work either. *Hint:* What pair does $\{\{1\}, \{2\}\}$ represent?

(c) Define

$$\text{pair}(a, b) ::= \{a, \{a, b\}\}.$$

Explain why representing (a, b) as $\text{pair}(a, b)$ uniquely determines a and b. *Hint:* Sets can't be indirect members of themselves: $a \in a$ never holds for any set a, and neither can $a \in b \in a$ hold for any b.

Extra practice with set formulas:

Problem 4.

A *formula of set theory* is a predicate formula that only uses the predicate "$x \in y$." The domain of discourse is the collection of sets, and "$x \in y$" is interpreted to mean the set x is one of the elements in the set y.

For example, since x and y are the same set iff they have the same members, here's how we can express equality of x and y with a formula of set theory:

$$(x = y) ::= \forall z. (z \in x \text{ IFF } z \in y).$$

Express each of the following assertions about sets by a formula of set theory. Expressions may use abbreviations introduced earlier (so it is now legal to use "$=$" because we just defined it).

(a) $x = \emptyset$.

(b) $x = \{y, z\}$.

(c) $x \subseteq y$. (x is a subset of y that might equal y.)

Now we can explain how to express "x is a proper subset of y" as a set theory formula using things we already know how to express. Namely, letting "$x \neq y$" abbreviate $\text{NOT}(x = y)$, the expression

$$(x \subseteq y \text{ AND } x \neq y),$$

describes a formula of set theory that means $x \subset y$.

From here on, feel free to use any previously expressed property in describing formulas for the following:

(d) $x = y \cup z$.

(e) $x = y - z$.

(f) $x = \text{pow}(y)$.

(g) $x = \bigcup_{z \in y} z$.

This means that y is supposed to be a collection of sets, and x is the union of all of them. A more concise notation for "$\bigcup_{z \in y} z$' is simply "$\bigcup y$."

Supplemental problem:

Problem 5.

For any set x, define $\text{next}(x)$ to be the set consisting of all the elements of x, along with x itself:

$$\text{next}(x) ::= x \cup \{x\}$$

Now we can define a sequence of sets $\nu_0, \nu_1, \nu_2, \ldots$ called the *finite ordinals* with a simple recursive recipe:

$$\nu_0 ::= \emptyset,$$
$$\nu_{n+1} ::= \text{next}(\nu_n).$$

So we have,

$$\nu_1 ::= \{\emptyset\}$$
$$\nu_2 ::= \{\emptyset, \{\emptyset\}\}$$
$$\nu_3 ::= \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$$

The finite ordinals are kind of weird, but have some engaging properties, and more important, they turn out to play a significant role in set theory.

(a) Prove that

$$\nu_{n+1} = \{\nu_0, \nu_1, \ldots, \nu_n\}. \tag{1}$$

(b) Conclude that $|\nu_n| = n$.

Hint: A set cannot be a member of itself.[4]

(c) Conclude that if μ, ν, ρ are finite ordinals and $\mu \in \nu \in \rho$, then $\mu \in \rho$. Likewise, if μ, ν are different finite ordinals, then $\nu \in \mu$ OR $\mu \in \nu$.

[4] By the Foundation Axiom, Section 8.3.2.