In-Class Problems Week 2, Wed.

Problem 1.
Let P be the proposition depending on propositional variable A, B, C, D whose truth values for each truth assignment to A, B, C, D are given in the table below. Write out both a disjunctive and a conjunctive normal form for P.

A	B	C	D	P
T	**T**	**T**	**T**	**T**
T	**T**	**T**	**F**	**F**
T	**T**	**F**	**T**	**T**
T	**T**	**F**	**F**	**F**
T	**F**	**T**	**T**	**T**
T	**F**	**T**	**F**	**T**
T	**F**	**F**	**T**	**T**
T	**F**	**F**	**F**	**T**
F	**T**	**T**	**T**	**T**
F	**T**	**T**	**F**	**F**
F	**T**	**F**	**T**	**T**
F	**T**	**F**	**F**	**F**
F	**F**	**T**	**T**	**F**
F	**F**	**T**	**F**	**F**
F	**F**	**F**	**T**	**T**
F	**F**	**F**	**F**	**T**

Hint: See Section 3.4.1.

Problem 2.
Use the equivalence axioms of Section 3.4.2 to convert the formula

$$A \text{ XOR } B \text{ XOR } C$$

to disjunctive—OR of AND's–form,
Hint: Start by replacing the XOR's with some AND's, OR's, and NOT's.

Problem 3.
A 3-conjunctive normal form (3CNF) formula is a conjunctive normal form (CNF) formula in which each OR-term is an OR of at most 3 *literals* (variables or negations of variables). Although it may be hard to tell if a propositional formula F is satisfiable, it is always easy to construct a formula $\mathcal{C}(F)$ that is

- a 3CNF,

- has at most 24 times as many occurrences of variables as F, and
- is satisfiable iff F is satisfiable.

Note that we do *not* expect $\mathcal{C}(F)$ to be *equivalent* to F. We do know how to convert any F into an equivalent CNF formula, and this equivalent CNF formula will certainly be satisfiable iff F is. But in many cases, the smallest CNF formula equivalent to F may be *exponentially larger* than F instead of only 24 times larger. Even worse, there may not be any **3**CNF equivalent to F.

To construct $\mathcal{C}(F)$, the idea is to introduce a different new variable for each operator that occurs in F. For example, if F was

$$((P \text{ XOR } Q) \text{ XOR } R) \text{ OR } (\overline{P} \text{ AND } S) \tag{1}$$

we might use new variables X_1, X_2, O and A corresponding to the operator occurrences as follows:

$$((P \underbrace{\text{XOR}}_{X_1} Q) \underbrace{\text{XOR}}_{X_2} R) \underbrace{\text{OR}}_{O} (\overline{P} \underbrace{\text{AND}}_{A} S).$$

Next we write a formula that constrains each new variable to have the same truth value as the subformula determined by its corresponding operator. For the example above, these constraining formulas would be

$$X_1 \text{ IFF } (P \text{ XOR } Q),$$
$$X_2 \text{ IFF } (X_1 \text{ XOR } R),$$
$$A \text{ IFF } (\overline{P} \text{ AND } S),$$
$$O \text{ IFF } (X_2 \text{ OR } A)$$

(a) Explain why the AND of the four constraining formulas above along with a fifth formula consisting of just the variable O will be satisfiable iff (1) is satisfiable.

(b) Explain why each constraining formula will be equivalent to a 3CNF formula with at most 24 occurrences of variables.

(c) Using the ideas illustrated in the previous parts, briefly explain how to construct $\mathcal{C}(F)$ for an arbitrary propositional formula F. (No need to fill in all the details for this part—a high-level description is fine.)

Problem 4.
Explain a simple way to obtain a conjunctive normal form (CNF) for a propositional formula directly from a disjunctive normal form (DNF) for its complement.

Hint: DeMorgan's Law does most of the job. Try working an illustrative example of your choice before describing the general case.