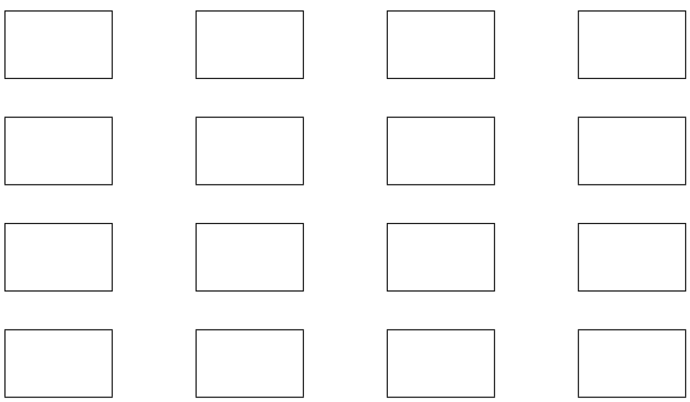
In-Class Problems Week 15, Mon.

Problem 1.
Here's a dice game with maximum payoff k: make three independent rolls of a fair die, and if you roll a six

- no times, then you lose 1 dollar;
- exactly once, then you win 1 dollar;
- exactly twice, then you win 2 dollars;
- all three times, then you win k dollars.

For what value of k is this game fair?[1]

Problem 2.
A novice mad scientist is attempting to put ions together into a salt crystal. The crystal has sixteen ion locations in a 4×4 arrangement as shown below.



If there is a sodium ion in front, behind, to the left, or to the right of a chloride ion, then the two *bond*. One ion may be in multiple bonded pairs; for example, an ion in a corner of the crystal can bond with up to two others, while an ion in the center can bond with as many as four others. Suppose that the locations are occupied mutually independently by sodium and chloride ions with equal probability. What is the expected number of bonded pairs? *Hint:* Linearity.

[1] This game is actually offered in casinos with $k = 3$, where it is called Carnival Dice.

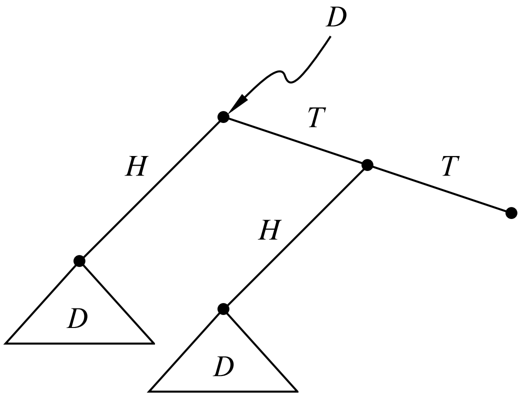


Figure 1 Sample space tree for coin toss until two consecutive tails.

Problem 3. **(a)** Suppose we flip a fair coin and let $N_{\texttt{TT}}$ be the number of flips until the first time two consecutive Tails appear. What is $\text{Ex}[N_{\texttt{TT}}]$?

Hint: Let D be the tree diagram for this process. Explain why D can be described by the tree in Figure 1. Use the **Law of Total Expectation**: Let R be a random variable and $A_1, A_2, \ldots,$ be a partition of the sample space. Then

$$\text{Ex}[R] = \sum_i \text{Ex}[R \mid A_i] \Pr[A_i].$$

(b) Let $N_{\texttt{TH}}$ be the number of flips until a Tail immediately followed by a Head comes up. What is $\text{Ex}[N_{\texttt{TH}}]$?

(c) Suppose we now play a game: flip a fair coin until either `TT` or `TH` occurs. You win if `TT` comes up first, and lose if `TH` comes up first. Since `TT` takes 50% longer on average to turn up, your opponent agrees that he has the advantage. So you tell him you're willing to play if you pay him \$5 when he wins, and he pays you with a mere 20% premium—that is \$6—when you win.

If you do this, you're sneakily taking advantage of your opponent's untrained intuition, since you've gotten him to agree to unfair odds. What is your expected profit per game?

Problem 4.
Let T be a positive integer valued random variable such that

$$\text{PDF}_T(n) = \frac{1}{an^2},$$

where

$$a ::= \sum_{n \in \mathbb{Z}^+} \frac{1}{n^2}.$$

(a) Prove that $\text{Ex}[T]$ is infinite.

(b) Prove that $\text{Ex}[\sqrt{T}]$ is finite.

Supplementary Problems

Problem 5.
Suppose there are 4 ions in a crystal, laid out in the corners of a square with corners 1, 2, 3 and 4.

Each corner is occupied by a sodium ion with probability $p > 0$ or a chloride ion with probability $q ::= 1 - p > 0$. A sodium ion and a chloride ion *bond* when they occupy adjacent corners of the square. Let $I_{12}, I_{23}, I_{34}, I_{41}$ be the indicator variables for a bonded pair being at the subscripted locations.

(a) What is the $\text{Ex}[I_{12}]$?

(b) What is the expected number of bonded pairs in terms of p and q?

(c) Prove that if $p = 1/2$ then the events $I_{12} = 1$ and $I_{23} = 1$ are independent.

(d) Prove conversely that if the events $I_{12} = 1$ and $I_{23} = 1$ are independent, then $p = 1/2$.

Problem 6.
Justify each line of the following proof that if R and S are *independent* random variables, then

$$\text{Ex}[R \cdot S] = \text{Ex}[R] \cdot \text{Ex}[S].$$

Proof.

$$\begin{aligned}
&\text{Ex}[R \cdot S] \\
&= \sum_{t \in \text{range}(R \cdot S)} t \cdot \Pr[R \cdot S = t] \\
&= \sum_{r \in \text{range}(R),\ s \in \text{range}(S)} rs \cdot \Pr[R = r \text{ and } S = s] \\
&= \sum_{r \in \text{range}(R)} \left(\sum_{s \in \text{range}(S)} rs \cdot \Pr[R = r \text{ and } S = s] \right) \\
&= \sum_{r \in \text{range}(R)} \left(\sum_{s \in \text{range}(S)} rs \cdot \Pr[R = r] \cdot \Pr[S = s] \right) \\
&= \sum_{r \in \text{range}(R)} \left(r \Pr[R = r] \cdot \sum_{s \in \text{range}(S)} s \Pr[S = s] \right) \\
&= \sum_{r \in \text{range}(R)} r \Pr[R = r] \cdot \text{Ex}[S] \\
&= \text{Ex}[S] \cdot \sum_{r \in \text{range}(R)} r \Pr[R = r] \\
&= \text{Ex}[S] \cdot \text{Ex}[R].
\end{aligned}$$

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