In-Class Problems Week 13, Wed.

Problem 1.
Solve the following problems using the pigeonhole principle. For each problem, try to identify the *pigeons*, the *pigeonholes*, and a *rule* assigning each pigeon to a pigeonhole.

(a) In a certain Institute of Technology, every ID number starts with a 9. Suppose that each of the 75 students in a class sums the nine digits of their ID number. Explain why two people must arrive at the same sum.

(b) In every set of 100 integers, there exist two whose difference is a multiple of 37.

(c) For any five points inside a unit square (not on the boundary), there are two points at distance *less than* $1/\sqrt{2}$.

(d) Show that if $n + 1$ numbers are selected from $\{1, 2, 3, \ldots, 2n\}$, two must be consecutive, that is, equal to k and $k + 1$ for some k.

Problem 2.
Section 15.8.6 explained why it is not possible to perform a four-card variant of the hidden-card magic trick with one card hidden. But the Magician and her Assistant are determined to find a way to make a trick like this work. They decide to change the rules slightly: instead of the Assistant lining up the three unhidden cards for the Magician to see, he will line up all four cards with one card face down and the other three visible. We'll call this the *face-down four-card trick*.

For example, suppose the audience members had selected the cards $9\heartsuit$, $10\diamondsuit$, $A\clubsuit$, $5\clubsuit$. Then the Assistant could choose to arrange the 4 cards in any order so long as one is face down and the others are visible. Two possibilities are:

$A\clubsuit$	?	$10\diamondsuit$	$5\clubsuit$
?	$5\clubsuit$	$9\heartsuit$	$10\diamondsuit$

(a) Explain how to model this face-down four-card trick as a matching problem, and show that there must be a bipartite matching which theoretically will allow the Magician and Assistant to perform the trick.

(b) There is actually a simple way to perform the face-down four-card trick.[1]

[1] This elegant method was devised in Fall '09 by student Katie E Everett.

Case 1. *there are two cards with the same suit*: Say there are two ♠ cards. The Assistant proceeds as in the original card trick: he puts one of the ♠ cards *face up as the first card*. He will place the second ♠ card *face down*. He then uses a permutation of the face down card and the remaining two face up cards to code the offset of the face down card from the first card.

Case 2. *all four cards have different suits*: Assign numbers $0, 1, 2, 3$ to the four suits in some agreed upon way. The Assistant computes s the sum modulo 4 of the ranks of the four cards, and chooses the card with suit s to be placed *face down as the first card*. He then uses a permutation of the remaining three face-up cards to code the rank of the face down card.

Explain how in Case 2. the Magician can determine the face down card from the cards the Assistant shows her.

(c) Explain how any method for performing the face-down four-card trick can be adapted to perform the regular (5-card hand, show 4 cards) with a 52-card deck consisting of the usual 52 cards along with a 53rd card called the *joker*.

Problem 3.

To ensure password security, a company requires their employees to choose a password. A length 10 word containing each of the characters:

a, d, e, f, i, l, o, p, r, s,

is called a *cword*. A password can be a cword which does not contain any of the subwords "fails", "failed", or "drop."

For example, the following two words are passwords: adefilopr s, srpolifeda,

but the following three cwords are not: a**drop**eflis, **failedrop**s, **drop**e**fails**.

(a) How many cwords contain the subword "drop"?

(b) How many cwords contain both "drop" and "fails"?

(c) Use the Inclusion-Exclusion Principle to find a simple arithmetic formula involving factorials for the number of passwords.

Problem 4.

How many paths are there from point $(0, 0)$ to $(50, 50)$ if each step along a path increments one coordinate and leaves the other unchanged? How many are there when there are impassable boulders sitting at points $(10, 11)$ and $(21, 20)$? (You do not have to calculate the number explicitly; your answer may be an expression involving binomial coefficients.)

Hint: Inclusion-Exclusion.

Optional Problem
(informative, but not on psets or exams)

Problem 5.

Let's develop a proof of the Inclusion-Exclusion formula using high school algebra.

(a) Most high school students will get freaked by the following formula, even though they actually know the rule it expresses. How would you explain it to them?

$$\prod_{i=1}^{n}(1 - x_i) = \sum_{I \subseteq \{1,\dots,n\}} (-1)^{|I|} \prod_{j \in I} x_j. \tag{1}$$

Hint: Show them an example.

Now to start proving (1), let M_S be the *membership* function for any set S:

$$M_S(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{if } x \notin S. \end{cases}$$

Let $S_1, \dots, S_n$ be a sequence of finite sets, and abbreviate M_{S_i} as M_i. Let the domain of discourse D be the union of the S_i's. That is, we let

$$D ::= \bigcup_{i=1}^{n} S_i,$$

and take complements with respect to D, that is,

$$\overline{T} ::= D - T,$$

for $T \subseteq D$.

(b) Verify that for $T \subseteq D$ and $I \subseteq [1..n]$

$$M_{\overline{T}} = 1 - M_T, \tag{2}$$

$$M_{(\bigcap_{i \in I} S_i)} = \prod_{i \in I} M_i, \tag{3}$$

$$M_{(\bigcup_{i \in I} S_i)} = 1 - \prod_{i \in I}(1 - M_i). \tag{4}$$

(Note that (3) holds when I is empty because, by convention, an empty product equals 1, and an empty intersection equals the domain of discourse D.)

(c) Use (1) and (4) to prove

$$M_D = \sum_{\emptyset \neq I \subseteq \{1,\dots,n\}} (-1)^{|I|+1} \prod_{j \in I} M_j. \tag{5}$$

(d) Prove that

$$|T| = \sum_{u \in D} M_T(u). \tag{6}$$

(e) Now use the previous parts to prove

$$|D| = \sum_{\emptyset \neq I \subseteq \{1,\dots,n\}} (-1)^{|I|+1} \left| \bigcap_{i \in I} S_i \right| \tag{7}$$

(f) Finally, explain why (7) immediately implies the usual form of the Inclusion-Exclusion Principle:

$$|D| = \sum_{i=1}^{n} (-1)^{i+1} \sum_{\substack{I \subseteq [1..n] \\ |I| = i}} \left| \bigcap_{j \in I} S_j \right|. \tag{8}$$