Mathematics for Computer Science
MIT 6.042J/18.062J

Combinatorial Proof

Pascal's Identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Algebraic Proof : routine, using

$$\binom{n}{k} ::= \frac{n!}{k!(n-k)!} = \frac{n(n-1)!}{k(k-1)!(n-k)!} = \frac{n}{k}\binom{n-1}{k-1}$$

Combinatorial Proof

classify subsets of $\{1,\ldots,n\}$

\# size k subsets =
\# size k subsets with 1
\+ # size k subsets without 1

Combinatorial Proof

classify subsets of $\{1,\ldots,n\}$

$$\underbrace{\binom{n}{k}}_{\#\text{ size k subsets}} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Combinatorial Proof

classify subsets of $\{1,\ldots,n\}$

$$\underbrace{\binom{n}{k}}_{\#\text{ size k subsets}} = \underbrace{\binom{n-1}{k}}_{\#\text{ size k subsets without 1}} + \binom{n-1}{k-1}$$

Combinatorial Proof

classify subsets of $\{1,\ldots,n\}$

QED

$$\underbrace{\binom{n}{k}}_{\#\text{ size k subsets}} = \underbrace{\binom{n-1}{k}}_{\#\text{ size k subsets without 1}} + \underbrace{\binom{n-1}{k-1}}_{\#\text{ size k subsets with 1}}$$

Combinatorial Proof, II

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$$

Combinatorial Proof, II

classify subsets of $\{1,\ldots,n,1,\ldots,n\}$

$$\text{RHS} = \underbrace{\binom{2n}{n}}_{\#\text{ size n subsets}}$$

Combinatorial Proof, II

$$LHS = \sum_{i=0}^{n} \binom{n}{i}^2$$

$$= \sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i}$$

Combinatorial Proof, II

LHS =

$$\sum_{i=0}^{n} \underbrace{\binom{n}{i}}_{\text{\# size i red subsets}} \binom{n}{n-i}$$

Combinatorial Proof, II

LHS =

$$\sum_{i=0}^{n} \underbrace{\binom{n}{i}}_{\text{\# size i red subsets}} \underbrace{\binom{n}{n-i}}_{\text{\# size n-i black subsets}}$$

Combinatorial Proof, II

$$\sum_{i=0}^{n} \underbrace{\binom{n}{i}}_{\text{\# size i red subsets}} \underbrace{\binom{n}{n-i}}_{\text{\# size n-i black subsets}}$$

So LHS = # size n subsets of {1,…,n,1,….,n} by Sum Rule

Combinatorial Proof, II

Therefore
LHS = # size n subsets = RHS

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$$

QED