

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

# Combinatorial Proof



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## Pascal's Identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Algebraic Proof : routine, using

$$\binom{n}{k} ::= \frac{n!}{k!(n-k)!} = \frac{n(n-1)!}{k(k-1)!(n-k)!} = \frac{n}{k} \binom{n-1}{k-1}$$



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## Combinatorial Proof

classify subsets of  $\{1, \dots, n\}$

# size  $k$  subsets =  
 # size  $k$  subsets with 1  
 + # size  $k$  subsets without 1



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## Combinatorial Proof

classify subsets of  $\{1, \dots, n\}$

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

# size  $k$  subsets



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## Combinatorial Proof

classify subsets of  $\{1, \dots, n\}$

$$\underbrace{\binom{n}{k}}_{\substack{\# \text{ size } k \\ \text{subsets}}} = \underbrace{\binom{n-1}{k}}_{\substack{\# \text{ size } k \\ \text{subsets} \\ \text{without } 1}} + \binom{n-1}{k-1}$$



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lec 11M.25

6	9	13	7
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## Combinatorial Proof

classify subsets of  $\{1, \dots, n\}$

$$\overset{\text{QED}}{\underbrace{\binom{n}{k}}_{\substack{\# \text{ size } k \\ \text{subsets}}}} = \underbrace{\binom{n-1}{k}}_{\substack{\# \text{ size } k \\ \text{subsets} \\ \text{without } 1}} + \underbrace{\binom{n-1}{k-1}}_{\substack{\# \text{ size } k \\ \text{subsets} \\ \text{with } 1}}$$



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6	9	13	7
12	10	5	
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15	8	11	2

## Combinatorial Proof, II

$$\sum_{i=0}^n \binom{n}{i}^2 = \binom{2n}{n}$$



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6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

## Combinatorial Proof, II

classify subsets of  $\{1, \dots, n, 1, \dots, n\}$

$$\text{RHS} = \underbrace{\binom{2n}{n}}_{\substack{\# \text{ size } n \\ \text{subsets}}}$$



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6	9	13	7
12		10	5
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## Combinatorial Proof, II

$$\begin{aligned} \text{LHS} &= \sum_{i=0}^n \binom{n}{i}^2 \\ &= \sum_{i=0}^n \binom{n}{i} \binom{n}{n-i} \end{aligned}$$



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6	9	13	7
12		10	5
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15	8	11	2

## Combinatorial Proof, II

LHS =

$$\sum_{i=0}^n \underbrace{\binom{n}{i}}_{\substack{\# \text{ size } i \\ \text{red subsets}}} \binom{n}{n-i}$$



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lec 11M.30

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Combinatorial Proof, II

LHS =

$$\sum_{i=0}^n \underbrace{\binom{n}{i}}_{\substack{\# \text{ size } i \\ \text{red subsets}}} \underbrace{\binom{n}{n-i}}_{\substack{\# \text{ size } n-i \\ \text{black subsets}}}$$



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lec 11M.31

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Combinatorial Proof, II

$$\sum_{i=0}^n \underbrace{\binom{n}{i}}_{\substack{\# \text{ size } i \\ \text{red subsets}}} \underbrace{\binom{n}{n-i}}_{\substack{\# \text{ size } n-i \\ \text{black subsets}}}$$

So LHS = # size n subsets of  $\{1, \dots, n, 1, \dots, n\}$  by Sum Rule



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lec 11M.32

6	9	13	7
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## Combinatorial Proof, II

Therefore

LHS = # size  $n$  subsets = RHS

$$\sum_{i=0}^n \binom{n}{i}^2 = \binom{2n}{n}$$

QED

