

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Mathematics for Computer Science

MIT 6.042J/18.062J

Cardinality (comparing the size of sets)



Albert R Meyer, March 4, 2015

cardinality.1

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Cantor's Idea

$A \text{ surj } B ::= \exists \text{ surj func: } A \rightarrow B$

think: " $|A| \geq |B|$ "

$A \text{ bij } B ::= \exists \text{ bijection: } A \rightarrow B$

think: " $|A| = |B|$ "



Albert R Meyer, March 4, 2015

cardinality.2

6	9	13	7
12		10	5
3	1	4	14
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$\text{pow}(\mathbb{N})$ bij ∞ -bit-strings

infinite set $\mathbb{N} = \{0, 1, 2, \dots\}$

subset: $\{0, \dots, 2, 3, \dots, 6, \dots\}$

string: 1 0 1 1 0 0 1 ...

a bijection from $\text{pow}(\mathbb{N})$ to
infinite bit-strings, $\{0, 1\}^\omega$



Albert R Meyer, March 4, 2015

cardinality.3

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Familiar "size" properties

$|A| = |B| = |C|$ IMPLIES $|A| = |C|$

$A \text{ bij } B \text{ bij } C$ IMPLIES $A \text{ bij } C$

proof:

have bijections $g: A \rightarrow B, f: B \rightarrow C$

need bijection $h: A \rightarrow C$

define $h ::= f \circ g$



Albert R Meyer, March 4, 2015

cardinality.4

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Familiar "size" properties

$|A| = |B| = |C|$ IMPLIES $|A| = |C|$
 $A \text{ bij } B \text{ bij } C$ IMPLIES $A \text{ bij } C$

$|A| \geq |B| \geq |C|$ IMPLIES $|A| \geq |C|$
 $A \text{ surj } B \text{ surj } C$ IMPLIES $A \text{ surj } C$



Albert R Meyer, March 4, 2015

cardinality.5

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Familiar "size" properties

$|A| \geq |B| \geq |A|$ IMPLIES $|A| = |B|$
 $A \text{ surj } B \text{ surj } A$ IMPLIES $A \text{ bij } B$

this is **NOT** obvious:
 Schroeder-Bernstein Thm



Albert R Meyer, March 4, 2015

cardinality.6

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

UNfamiliar "size" property

"size + 1 = size"
 for ∞ -sizes



Albert R Meyer, March 4, 2015

cardinality.7

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Same Size Infinite Sets?

$\{1, 2, 3, 4, 5, \dots\}$

↑ ↑ ↑ ↑ ↑

$\{0, 1, 2, 3, 4, \dots\}$

\mathbb{N} "same size" \mathbb{Z}^+ !



Albert R Meyer, March 4, 2015

cardinality.8

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Same Size Infinite Sets?

$\{0, 1, -1, 2, -2, \dots\}$



$\{0, 1, 2, 3, 4, \dots\}$

\mathbb{N} "same size as" \mathbb{Z}

