Mathematics for Computer Science

MIT 6.042J/18.062J

Uncountable Sets

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Infinite Sizes

Are all sets the same size? NO!

Cantor's Theorem

shows how to keep finding bigger infinities.

Countable Sets

A is countable iff can list it:

$a_0, a_1, a_2, \ldots.$ example:

$\{0,1\}^*$ ::= {finite bit strings}

Claim: $\{0,1\}^\omega$ ::= {∞-bit strings} is uncountable.

Diagonal Arguments

Suppose $s_0, s_1, s_2, \ldots \in \{0,1\}^\omega$

	0	1	2	3	.	.	.	n	n+1	.	.	.
s_0	0	0	1	0	.	.	.	0	0	.	.	.
s_1	0	1	1	0	.	.	.	0	1	.	.	.
s_2	1	0	0	0	.	.	.	1	0	.	.	.
s_3	1	0	1	1	.	.	.	1	1	.	.	.
.	.	.			1							
.	.		.			1						
	.			.			0					

Diagonal Arguments

Suppose $s_0, s_1, s_2, \ldots \in \{0,1\}^{\omega}$

	0	1	2	3	.	.	.	n	n+1	.	.	.
s_0	1	0	1	0	.	.	.	0	0	.	.	.
s_1	0	0	1	0	.	.	.	0	1	.	.	.
s_2	1	0	1	0	.	.	.	1	0	.	.	.
s_3	1	0	1	0	.	.	.	1	1	.	.	.
.	.	.			0							
.	.		.			0						
	.			.			1					

Diagonal Arguments

Suppose $s_0, s_1, s_2, \ldots \in \{0,1\}^{\omega}$

1 ...
0
1
0
0
0
1

Diagonal Arguments

Suppose $s_0, s_1, s_2, \ldots \in \{0,1\}^{\omega}$

1 0 1 0 0 0 1 ...

Diagonal Arguments

So 1 0 1 0 0 0 1... differs from every row.

Diagonal Arguments

So $1\ 0\ 1\ 0\ 0\ 0\ 1\cdots$ differs from every row. That is,

$$1\ 0\ 1\ 0\ 0\ 0\ 1\cdots \notin \{0,1\}^{\omega}$$

contradicting the claim that every $s \in \{0,1\}^{\omega}$ appears as a row.

$\{0,1\}^{\omega}$ is uncountable

So NOT $\left(\mathbb{N} \text{ surj } \{0,1\}^{\omega}\right)$

$\mathbb{N}$ "strictly smaller" than $\{0,1\}^{\omega}$

Strictly Smaller

A strict B ::= NOT(A surj B)

A is “strictly smaller” than B

So $\mathbb{N}$ strict $\{0,1\}^{\omega}$

Strictly Smaller

Warning!

“Strictly smaller” is a metaphor

Strictly Smaller

Warning!

"Strictly smaller" is a metaphor meant to suggest properties of the highly technical relation strict.

Strictly Smaller

Warning!

We do not assign "sizes" to infinite sets.

Informal reasoning about "strictly smaller" will be misleading for strict.

Cantor's Theorem

A strict $pow(A)$

for every set, A

(finite or infinite)

Diagonal Arguments

Suppose $A = \{a, b, s, t, \ldots, d, e, \ldots\}$

$pow(A) = \{f(a), f(b), f(s), \ldots, f(d), \ldots\}$

	a	b	s	t	.	.	.	d	e	.	.	.
f(a)												.
f(b)												.
f(s)												.
f(t)												.
.												

Diagonal Arguments

Suppose $A = \{a, b, s, t, \ldots, d, e, \ldots\}$

$pow(A) = \{f(a), f(b), f(s), \ldots, f(d), \ldots\}$

	a	b	s	t	c	.	.	d	e	.	.	.
f(a)	a		s	t					e			.
f(b)	a	b			c			d				.
f(s)		b		t								.
f(t)			s	t	c			d				.
f(c)		b	s					d	e			
.												
.												

Diagonal Arguments

Suppose $A = \{a, b, s, t, \ldots, d, e, \ldots\}$

$pow(A) = \{f(a), f(b), f(s), \ldots, f(d), \ldots\}$

	a	b	s	t	c	.	.	d	e	.	.	.
f(a)	~~a~~		s	t					e			.
f(b)	a	~~b~~			c			d				.
f(s)		b	s	t								.
f(t)			s	~~t~~	c			d				.
f(c)		b	s		c			d	e			
.						.						
.							.					

Diagonal Arguments

Suppose $A = \{a, b, s, t, \ldots, d, e, \ldots\}$

$pow(A) = \{f(a), f(b), f(s), \ldots, f(d), \ldots\}$

D	a	b	s	t	c	.	.	d	e	.	.	.
f(a)			s	t					e			.
f(b)	a				c			d				.
f(s)		b	s	t								.
f(t)			s		c			d				.
f(c)		b	s		c			d	e			
.						.						
.							.					

A strict Pow(A)

Pf: say have fcn $f: A \to pow(A)$.

Define a subset of A that is not in the range of f: namely

$$D ::= \{a \in A \mid a \notin f(a)\}$$

Now $D \notin range(f)$ since it differs from set $f(a)$ at element a!

A strict Pow(A)

So no f-arrow into D.

f is not a surjection.

QED

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ℕ strict pow(ℕ)

That is,

pow(ℕ) is uncountable

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Proving Uncountability

Lemma: If A is uncountable and C surj A then C is uncountable

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$\{0,1\}^\omega$ again

We know

$\{0,1\}^\omega$ bij pow(ℕ)

and pow(ℕ) uncountable by Cantor, so $\{0,1\}^\omega$ uncountable.

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Real Numbers Uncountable

$\mathbb{R}$ surj $\{0,1\}^{\omega}$

map $\pm r$ to binary rep

7 1/3 = 111.010101…

maps to 111010101…

Real Numbers Uncountable

$\mathbb{R}$ surj $\{0,1\}^{\omega}$

map $\pm r$ to binary rep

1/2 = .100000…

1/2 maps to 100000…

Real Numbers Uncountable

$\mathbb{R}$ surj $\{0,1\}^{\omega}$

map $\pm r$ to binary rep

1/2 = .100000…

1/2 maps to 100000…

= .0111111…

-1/2 maps to 0111111…