

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Mathematics for Computer Science  
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# Bijections for Counting



Albert R Meyer, April 17, 2013

bijectcount.1

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## Example: Counting Passwords

Password conditions:

- characters are digits & letters
- between 6 & 8 characters long
- starts with a letter
- case sensitive



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bijectcount.2

6	9	13	7
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## Counting Passwords

$L ::= \{a, b, \dots, z, A, B, \dots, Z\}$

$D ::= \{0, 1, \dots, 9\}$

$P_n ::=$  length  $n$  words  
starting w/letter

$$= L \times (L \cup D)^{n-1}$$



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bijectcount.3

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## Counting Passwords

$$\begin{aligned} & |L \times (L \cup D)^{n-1}| \\ &= |L| \cdot |(L \cup D)|^{n-1} \\ &= |L| \cdot (|L| + |D|)^{n-1} \\ &= 52 \cdot (52 + 10)^{n-1} \end{aligned}$$



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bijectcount.4

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Counting Passwords  
set of passwords:

$$P ::= P_6 \cup P_7 \cup P_8$$

$$|P| = |P_6| + |P_7| + |P_8|$$

$$= 52 \cdot (62^5 + 62^6 + 62^7)$$

$$\approx 19 \cdot 10^{14}$$



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bijectcount.5

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# 4-digit nums w/  $\geq$  one 7  
cases by 1st occurrence of 7:  
x: any digit    o: any digit  $\neq$  7  
7xxx or o7xx or oo7x or ooo7  
 $10^3 + 9 \cdot 10^2 + 9^2 \cdot 10 + 9^3$   
 $= 3439$



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bijectcount.6

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at least one 7: another way  
 $|4\text{-digit nums w/ } \geq \text{one } 7|$   
 $= |4\text{-digit nums}|$   
 $\quad - |those w/ no 7|$   
 $= 10^4 - 9^4 = 3439$



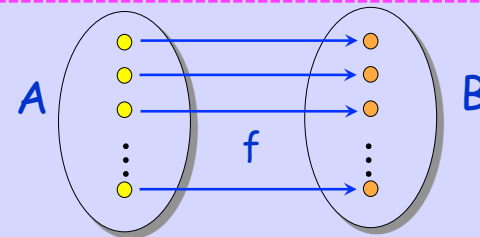
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bijectcount.7

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Mapping Rule: Bijections

If  $f$  is a bijection from  $A$  to  $B$ ,  
then  $|A| = |B|$



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bijectcount.8

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## Bijection: $\mathcal{P}(A)$ and Binary Strings

$A: \{a_1, a_2, a_3, a_4, a_5, \dots, a_n\}$

subset:  $\{a_1, a_3, a_4, \dots, a_n\}$

string: 1 0 1 1 0 ... 1

This is a bijection, so

$$\underbrace{|\text{n-bit binary strings}|}_{2^n} = |\mathcal{P}(A)|$$

$2^n$



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bijectcount.10

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## Size of $\mathcal{P}(A)$

$$|\mathcal{P}(A)| = 2^{|A|}$$



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bijectcount.11

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## Counting Doughnut Selections

From 5 kinds of doughnuts  
select a dozen.

let  $A ::=$  all selections of  
12 doughnuts

$\underbrace{00}_{\text{chocolate}}$ 
 $\underbrace{(none)}_{\text{lemon}}$ 
 $\underbrace{000000}_{\text{sugar}}$ 
 $\underbrace{00}_{\text{glazed}}$ 
 $\underbrace{00}_{\text{plain}}$



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bijectcount.12

6	9	13	7
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## Counting Doughnut Selections

$B ::=$  16-bit words with four 1's

0011000000100100

$\underbrace{001}_{\text{chocolate}}$ 
 $\underbrace{1000000}_{\text{sugar}}$ 
 $\underbrace{100}_{\text{glazed}}$ 
 $\underbrace{100}_{\text{plain}}$



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bijectcount.15

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## Bijection from $A$ to $B$

$c$  chocolate,  $l$  lemon,  $s$  sugar,  $g$  glazed,  $p$  plain  
maps to

$0^c 10^l 10^s 10^g 10^p$

so

$$|A| = |B|$$



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April 17, 2013

bijecount.16