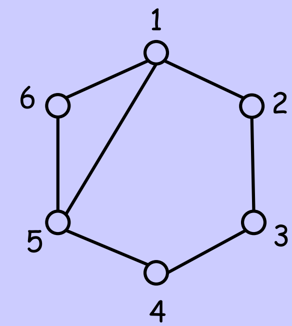
Mathematics for Computer Science

MIT 6.042J/18.062J

3-Coloring Reduces to SAT

A Simple Graph G



Propositional Variables

Want a formula that means

G is 3-colorable

Do this using variables

R_1 W_1 B_1

which will mean that vertex 1 is colored Red, White, or Blue

Propositional Formula

$[R_1$ OR W_1 OR $B_1]$

AND

which will mean that vertex 1 is colored Red, White, or Blue

Propositional Formula

$[(R_1 \text{ AND } \overline{W_1} \text{ AND } \overline{B_1}) \text{ OR}$

$(W_1 \text{ AND } \overline{R_1} \text{ AND } \overline{B_1}) \text{ OR}$

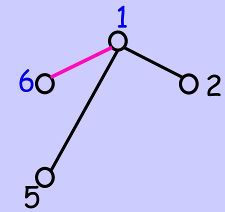
$(B_1 \text{ AND } \overline{W_1} \text{ AND } \overline{R_1})]$

and vertex 1 has

only one color

Propositional Formula

AND vertex 1 has a different color than vertex 6



$\text{NOT}(R_1 \text{ AND } R_6) \text{ AND}$

$\text{NOT}(W_1 \text{ AND } W_6) \text{ AND}$

$\text{NOT}(B_1 \text{ AND } B_6)$

Propositional Formula

Do the same for vertices 2-6

and for the remaining six edges.

Let

$$\text{Prop}_G$$

be the AND of all these formulas

SAT vs 3-Color

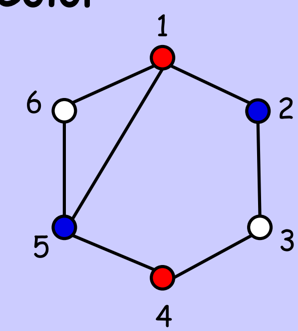
G is 3-colorable

so setting

$\underbrace{R_1 R_4 W_3 W_6 B_2 B_5}_{\text{True}}$

and the rest False

satisfies Prop_G



SAT vs 3-Color

For any simple graph G construct $Prop_G$ similarly.

G is 3-colorable iff

$Prop_G$ is satisfiable

SAT vs 3-Color

3-Color reduces to SAT:

a good SAT procedure would yield a good 3-Coloring procedure (and conversely)

SAT vs 3-Color

SAT and 3-Color stand and fall together: there is an "efficient" (polynomial time) SAT procedure iff there is one for 3-Color. Both problems are NP-complete.